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EDITORS

Ceneida Fernández / Salvador Llinares
Ángel Gutiérrez / Núria Planas

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Research Reports (A – H)

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RESEARCH REPORTS (A-H)

ASPECTS OF MATHEMATICAL KNOWLEDGE FOR TEACHING: A QUALITATIVE STUDY

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We present a qualitative study aiming at investigating secondary school teachers' Mathematical Knowledge for Teaching regarding the dense ordering of rational numbers. Fifteen secondary math teachers were asked to evaluate the responses of hypothetical students, explain students' thinking, and give feedback. The accuracy of the evaluation, the quality of the explanation, and the use of counterexamples were examined. The results showed shortcomings in various categories of Mathematical Knowledge for Teaching, such as Common Content Knowledge and Specialized Content Knowledge, Knowledge of Content and Students and Knowledge of Content and Teaching.

THEORETICAL FRAMEWORK

One of the major concerns in educational research is the knowledge required for teaching. Ball and her colleagues (Ball, Thames & Phelps, 2008) outlined certain components of Mathematical Knowledge for Teaching that have been a reference point for mathematics education researchers. In this paper, we adopted Ball and colleague's theoretical framework and we studied aspects of the Mathematical Knowledge for Teaching in secondary school math teachers.

One of the aspects of Mathematical Knowledge for Teaching that we focused on is Common Content Knowledge, which is knowledge about the mathematical content that is useful for teaching, albeit not exclusively. The second aspect of Mathematical Knowledge for Teaching we are interested in is Knowledge of Content and Students, in terms of teachers' ability to explain students' thinking, especially when they give incorrect answers. We focused on the use of counterexamples, which relates to Specialized Content Knowledge (i.e., knowledge which is useful exclusively for teaching) and Knowledge of Content and Teaching. Indeed, the appropriate selection and use of counterexamples in teaching is a very important, non-trivial process (Zaslavsky, 2010). The fundamental purpose of a counterexample is to refute a claim. In teaching, however, appropriate selection and use of counterexamples is required to make visible to students the reasons why a claim is false and to create conditions for generalization, beyond the particular claim; these are found to be challenging for teachers (Pele & Zaslavksy, 1997; Zaslavsky, 2010).

The study presented in this paper is part of a larger one investigating secondary math teachers' Mathematical Knowledge for Teaching about rational and real numbers. Here we focus on Mathematical Knowledge for Teaching about the dense ordering of rational numbers. It is amply documented that this property is difficult for students at

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all levels of education, even for tertiary students studying mathematics. Students often argue that between two rational numbers there is a finite, possibly zero number of numbers, as it happens in the set of national numbers. Moreover, even students who describe the number of intermediate numbers as “infinite” often refer to a very large number (e.g., “a billion”, “as many as the grains of sand in the desert”) that it is finite (Vamvakoussi & Vosniadou, 2012).

Our research questions were: a) Do teachers evaluate correctly students’ answers about the number of numbers in an interval? (Common Content Knowledge) b) Are teachers able to explain the students’ way of thinking? (Knowledge of Content and Students) and c) What are the characteristics of the feedback they give to the wrong answers? Specifically, for (c), we examined teachers’ selection and use of counterexamples (Specialized Content Knowledge, Knowledge of Content and Teaching).

METHOD

Participants

The participants were 15 secondary math teachers (10 women, 5 men) with 1 to 7 years of teaching experience. In Greece, secondary math teachers necessarily have a degree in mathematics. The majority of our participants were either in the process of obtaining or had already obtained a master’s degree. One of them had a master’s degree in Mathematics Education.

Research tool

To explore teachers’ Mathematical Knowledge for Teaching about the dense ordering of rational numbers we used tasks in the form of (hypothetical) classroom scenarios, deemed suitable for such purposes (Biza, Nardi & Zachariades, 2007). Due to space limitations, in this paper we will examine only one of them in detail.

According to this classroom scenario, a hypothetical teacher asks a class of 9^{th} graders how many numbers there are between 1.1 and 1.3; three different responses (A, B, C) by the students are presented: A) “One, 1.2”; B) “19: 1.11, 1.12, 1.13, 1.14, 1.15, .., 1.19, 1.20, 1.21, .., 1.29”; and C) “They are infinite… lots of them… over a billion. Only a computer could find them all.” The three hypothetical responses A, B and C correspond to different levels of understanding of the number of intermediate numbers in an interval (Vamvakoussi & Vosniadou, 2012). In A, the student treats the given numbers as natural numbers. In B, the hypothetical student performs the first step of a potentially repeatable process by adding a decimal digit to the given numbers (1.10 and 1.30) but then treats them similarly to student A. Finally, answer C corresponds to the interpretation of the expression “infinity numbers” as “a very large, but finite, number of numbers”.

The participants were asked the following questions: a) Is any of these answers, correct? If so, which one? If not, which is the correct answer?, b) Can you explain each student’s thinking?, and c) How would you deal with this situation, if you were the teacher of this class? How would you give feedback to these students?

Procedure

Teachers participated individually in semi-structured interviews, which were conducted via Skype. All the interviews were recorded and transcribed.

RESULTS

Evaluating and explaining students' answers

We first examined whether the teachers evaluated correctly the three responses (A, B, C). Nine of the fifteen teachers correctly evaluated all three answers. While all of them correctly judged that answers A and B were incorrect, six teachers (T2, T3, T6, T7, T11, T13) considered answer C to be correct. For example:

> T7: I agree that a computer could find them all. If it were programmed by a mathematician, the computer would run the algorithm and find them all.

Explanations of the student's thinking were examined in cases where the assessment was correct and were categorized into 3 categories. The first category ("No Explanation") included responses in which participants either explicitly said that they were unable to explain; or avoided giving an explanation. The second category ("Trivial explanation") included all explanations that repeated the student's answer, or attributed the error to general factors such as the student's background in mathematics ("strong"/ "weak" student), or carelessness. The following extracts present examples of the first (T2) and second (T1) category of explanations:

> T2: Now, how did he come up with it? I don't know how he thought of it.

> T1: It can be due to a number of factors. This student might be weak, or careless.

Finally, the category "Relevant Explanation" included the explanations that provided a substantial rationale for the hypothetical students' thinking. For A and B, teachers who gave relevant explanations appeared to recognize that the students' reasoning was based on natural number knowledge (see T15 in the excerpt below). For C, teachers who provided relevant explanations appeared to acknowledge that the hypothetical student, while using the term "infinity", was actually referring to a very large, albeit finite, number of intermediate numbers (see T15 in the following excerpt).

> T5: Well, the first one thought that after 1.1… in the decimal part, after 1 there's 2 and then 3. So, between 1.1 and 1.3 there is 1.2.

> T15: The third one says there are infinitely many, but the fact that he says there are over a billion, he puts a barrier, he is, like, counting them. I don't think he understands what infinity is.

Table 1 shows the frequency of each category of explanation by hypothetical answer. Less than half of the explanations were found in the "Relevant Explanation" category, with the majority of them for A. We should note that only three teachers gave relevant explanations for all three hypothetical answers (T4, T8, T15). In addition, 5 participants didn't provide relevant explanations for any of the responses that they had assessed correctly, including two who had assessed all three correctly (T1, T14).

Explanation Category	Hypothetical Answers			Total
	A	B	C	
Relevant Explanation	9	4	5	18
Trivial Explanation	5	7	2	14
No Explanation	1	4	2	7
Total	15	15	9	39

Table 1: Frequency of each category of explanation by hypothetical answer.

It is also interesting to note that 5 of the teachers who gave a substantial explanation for A didn't recognize that B was a similar case (Table 1). One such example is T10:

T10: The second answer is a bit strange, it's weird. Uh… (pause). I don't know. I don't know where this answer comes from, I really can't imagine.

Feedback: The use of counterexamples

We analyzed the teachers' feedback to the hypothetical students only for the responses they had (correctly) assessed to be incorrect. We note that in many cases the teachers addressed more than one response simultaneously. In the relevant texts, we searched for references to counterexamples, initially individually. We found that there were direct and indirect such references, so the texts were reviewed, the findings were compared and the (few) differences were resolved by discussion.

In total, 14 references to counterexamples were identified. Counterexamples were mentioned explicitly (as specific numbers) or descriptively (e.g., "decimals with many decimal digits"); they were also implied via referring to the responses of other hypothetical students or to a modified form of the problem in which more intermediate numbers were considered to be visible to students (e.g., after adding one or more zeros to the decimal part of the given numbers). Texts containing references to counterexamples were first examined as to whether counterexamples are used merely to refute a particular claim or whether their use afforded possibilities for generalization. Two initial categories were created.

The first category ("Claim Refutation", N=4) included cases in which the teacher referred to one or more intermediate numbers, with the intention of refuting the claim that "there are no other intermediate numbers". For example:

T15: I would ask them, is—let's say—1.135 between these numbers? I think that all the students would say "yes, this is in between". Then I would ask, is 1.1355 between? It is. That's how they would understand that they were wrong.

The second category ("Potential Generalization", N=7) included the cases that referred to a method of generating counterexamples that potentially leads to the infinity of the intermediate numbers in an interval. However, differences were found in the adequacy of the description of the method, and two subcategories were formed. In all cases of the first category ("Potential Generalization – Inadequate description, N= 4), teachers were limited to mentioning the possibility of more/fewer decimal digits in a number, similar to T3 in the extract below:

> T3: I would explain to them that after the decimal point, we can put infinitely many digits, that's a number too. The number 1.113758239 is smaller than 1.2.

In the second subcategory ["Potential Generalization – Adequate Description", N=3], we included cases in which teachers explicitly described a generalizable, repeatable process of generating intermediate numbers, either in a purely numerical context (T1, T10), or in the context of the number line (T12). For example:

> T10: At first, I would pay attention to the first two answers (*A and B*). I would say to the students: the first step is simply to say 1.2. But then, as the other student said, we can take a second step and add two decimal digits, 1.12, 1.13, 1.29. I would say to them, if we got one decimal digit the first time and two decimal digits the second time, why don't we continue to three decimal digits? I would then say that what I'm telling you now, we can do for 4 decimal digits as well. So, it's a process that we can keep doing for any number of decimal digits. Since we can do this for any number of digits, we begin to understand that there are infinitely many numbers in between.

> T12: I will tell them to pick any two numbers on the number line. I'm going to take the point in the middle of the line segment. So, here is a number in between. Then, I will pick one of the two (*endpoints*) and I will do the same. We can zoom in again and again and find infinitely many numbers.

We note that T1 and T12 also expressed a clear intention to address the infinity of intermediate numbers in any interval.

Finally, in the "Other" category (N= 3), three cases of feedback using counterexamples were included, which were judged, for different reasons, as inappropriate (see excerpts below). More specifically, T13 relied on an invalid argument, claiming that since all real numbers are infinitely many, there are infinitely many numbers in any interval. T7's feedback had two parts. In the first, she stated that there are infinitely many numbers in the interval, referring to infinity as an "unending process". In the second, she described vaguely the intermediate numbers as "numbers with "lots of decimal digits". No obvious connection between the two parts was made; and it is unclear how the intermediate numbers are generated, and whether "lots of" is also used to refer to "infinitely many". Finally, T14 based all the counterexamples as well as their generation method, on the sequence of natural numbers, referring to the first four terms of the corresponding sequence of decimal numbers with one decimal digit. It is not clear which number follows 0.9 in his sequence. Assuming that it is 1, then the subsequent terms are not between 0 and 1. Assuming that it is 0.10, then this sequence is presented with the misleading ordering of natural numbers, consistent with the well-known misconception that "longer decimals are larger"

> T13: Because these numbers can be placed on the real line, and because there are infinitely many real numbers, it is obvious that between two numbers there isn't just one, two etc., there are lots of numbers, which are not easy to find.

Usually, in our everyday life and when we teach, we use "easier" numbers such as 1.13, 1.14 etc.

T7: I would tell them that there are infinitely many numbers between 1.1 and 1.3 and we can't say precisely how much "infinitely many" is, infinity means you keep going. Anyway, there will be numbers with lots of decimal digits that approach 1.3.

T14: The answer is the same for all, so I would tell them that the numbers, as we know them, are infinitely many. (…). When I go from 0 to 1, there are also infinitely many numbers. You see, when I count 0, 1, 2, 3, 4, 5, I can continue to infinity. If I want to go from 0 to go to 1, there is the number 0.1. So, I can go on, 0.2, 0.3 up to infinity, just like before. Just like I reached infinity the first time, I also reach infinity by 0.1, 0.2, 0.3, 0.4, there are infinitely many numbers up to 1.

Finally, we would like to highlight another aspect of teachers' feedback which we had not anticipated and emerged through their descriptions. As can be noticed from the preceding excerpts, the students are hardly taken into account. We reviewed the texts for indications of intention to include students in the process in a meaningful way (i.e., intention to ask students to explain their answers, to compare answers, to explore non-trivial questions, etc.). Only two teachers (T1, T6) expressed the intention to engage the students in the process. For example:

T1: I would start with the second student and the rationale of the third so that we can come to the conclusion that no matter what interval we end up taking, we will always find an intermediate number (….) I generally prefer in such cases to ask the children to explain their peers' mistakes. In this way we eventually end up with the correct answer.

DISCUSSION

In this paper we examined aspects of Mathematical Knowledge for Teaching (Ball et al., 2008) of secondary school teachers. We focused on participants' responses to hypothetical students' answers to a question about the number of intermediate numbers in a given interval. The first finding, which relates to Common Content Knowledge (Ball et al., 2008), was that only 9 out of 15 teachers evaluated correctly all three hypothetical responses. This is remarkable given the mathematical background of the particular participants. The remaining 6 teachers agreed with the claim that "a computer can find all the intermediate numbers in a given interval". This reflects a conception of "infinitely many" as "a very large, albeit finite number", similar to the conception documented for primary and secondary school students (Vamvakoussi & Vosniadou, 2012). Regarding Common Content Knowledge, we should also note the invalid argument presented by one participant in the attempt to give feedback to the hypothetical students ("there are infinitely many real numbers, therefore there are infinitely many intermediates in the given interval").

Explaining the hypothetical students' thinking (Knowledge of Content and Students; Ball et al., 2008) also proved challenging. A meaningful explanation, acknowledging implicitly or explicitly that decimal numbers were treated by the first two hypothetical students like natural numbers, was provided by the majority of participants in the simpler case (i.e., “only 1.2 between 1.1 and 1.3”), but by fewer in a similar one (i.e., “only 1.11, 1.12, …1.29 between 1.1 and 1.3”). Only five teachers recognized the misinterpretation of the expression “infinitely many” by the third student. In the majority of cases, no explanation or a trivial explanation was given, attributing the error to factors such as carelessness, or the general mathematical background of the student, which is not conducive to meaningful instructional support for students. We note that accurate evaluations did not necessarily imply relevant explanations, which highlights the fact that Common Content Knowledge is distinct from Knowledge of Content and Students.

Finally, in terms of feedback, we focused on the use of counterexamples, which relates to Specialized Content Knowledge as well as to Knowledge of Content and Teaching (Ball et al., 2008). The particular classroom scenario afforded the use of counterexamples and the participants indeed used them; they also had methods of producing counterexamples at their disposal. Only three teachers, however, placed the counterexamples in an explanatory context that could support students to understand the underlying method and reach the conclusion that there are infinitely many numbers in any interval. This is consistent with findings showing that the appropriate use of counterexamples is challenging for teachers (Zaslavsky, 2010).

Finally, with only two exceptions, the teachers did not explicitly express the intention to engage the students in any productive activity during the feedback process. They typically described a situation where the teacher presents and explains the correct answer; and the students’ participation is minimal, if not trivial.

The findings on feedback, although they give some (alarming) indications regarding the teachers’ ways of dealing with similar situations in the classroom, should be treated with caution. Indeed, it is possible that in real classroom settings the teachers would have engaged in a more meaningful interaction with students or presented more elaborate explanations.

To sum up, the results of this study indicated shortcomings in the aspects of Mathematical Knowledge for Teaching that were investigated. The findings can’t be generalized, due to the small sample size, but can be used as a starting point for deeper exploration of these issues in the future.

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