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Commentary

The development of rational number knowledge: Old topic, new insights

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ABSTRACT

The development of rational number knowledge has been studied extensively by mathematics education researchers, cognitive-developmental psychologists and, more recently, by neuroscientists as well. Building on a rich body of prior research, and some exciting new ideas, the target articles re-visit several topics, with a view to refine and deepen our understanding of how rational number understanding develops. The effect of prior natural number knowledge-either positive or adverse-on rational number learning is highlighted by all contributors. I draw on the articles to discuss five different aspects of the whole, or natural, number bias.

1. Introduction

This special issue addresses the development of rational number knowledge,[1] an issue that has been studied extensively by mathematics education researchers, cognitive-developmental psychologists and, more recently, has attracted the interest of neuroscientists as well (e.g., Jacob, Vallentin, & Nieder, 2012). Building on a rich body of prior research, and some exciting new ideas, the contributors re-visit several topics, with a view to refine and deepen our understanding of how rational number understanding is developed. Kelley and Rittle-Johnson study misconceptions about decimal numbers in connection to the individual's confidence about the response; McMullen, Laakkonen, Hannula-Sormunen, and Lehtinen study the development of students' understanding of density in a longitudinal study and with the use of new statistical techniques; Van Hoof, Vandewalle, Verschaffel, and Van Dooren take a closer look at students' interpretation of literal symbols and their understanding of the effect of operations combined; DeWolf and Vosniadou, and Torbeyns, Schneider, and Siegler look into fraction magnitude representations with a view to support two different, albeit not incompatible, theoretical views.

2. Rational numbers are difficult … but why?

Let me start this commentary with a point stressed in all contributions, namely that rational numbers are challenging for students. Drawing on empirical evidence as well as conceptual analyses coming for numerous studies, Moss (2005) summarizes several reasons why rational numbers are difficult: Students need to construct a complex knowledge network based on multiplicative rather than on additive relations; new symbols and representations are introduced that need to be understood and coordinated; the notion of the unit and of the arithmetical operations need to be reconceptualised; and there are several conceptually distinct meanings attached to rational numbers that, again, need to be understood and coordinated. These include the part-whole aspect of fraction, fraction as a quotient, fraction as a multiplicative operator, fraction as a ratio, and fraction as measure. The latter is closely related to an aspect that is particularly relevant to this special issue: Rational numbers are *numbers*, that is, abstract entities that take their meaning within a number system, through their relations with other numbers and in accordance with certain principles and rules, such as the basic laws of arithmetic (e.g., Kilpatrick, Swafford, & Findell, 2001). There is a huge difference between abstract and concrete conceptualizations of number (both historically and developmentally). Indeed, "three apples are more than two apples" is not the same as "3 is bigger than 2"; similarly, "half of an apple is more than one quarter of the apple" is not the same as "1/2 is bigger than 1/4". The latter, abstract, conceptualization is far more challenging (Kilpatrick et al., 2001; Ni & Zhou, 2005). All five articles of this special issue address precisely this

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[1] For the purposes of this commentary, the term rational number is used to refer to numbers that are in, or can be converted to, the form a/b, where a and b are integers and b is non-zero. This is consistent with the school mathematics definition of the rational numbers, and by no means equivalent to the formal definition of the rational numbers.

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abstract conceptualization of number: Participants are asked to compare decimals (Kelley & Rittle-Johnson) and fractions (DeWolf & Vosniadou, McMullen et al., 2015; Torbeyns et al., 2015); and place fractions on the number line (Torbeyns et al., 2015). It is only in this abstract context that numbers can be understood as densely ordered (McMullen et al., 2015). Furthermore, this kind of abstraction is required in order to eventually conceptualize rational numbers as a unified number system (Kilpatrick et al., 2001), which is necessary in order to assign more than one type of numbers to real variables and judge the effect of operations with unknown numbers (Van Hoof et al., 2015).

Moss's (2005) summary illustrates nicely two main points that are relevant to the articles of this special issue: Rational numbers are difficult for students, because a lot of new material has to be learnt, and the content is highly complex (even without considering the vast variety of related applications). Furthermore, prior knowledge and experience with natural numbers is not always supportive of rational number learning, a phenomenon noticed, studied, and reported by many mathematics education researchers (see the studies cited in Kilpatrick et al., 2001; Moss, 2005; Ni & Zhou, 2005; Vamvakoussi, Vosniadou, & Van Dooren, 2013), far before the term “whole number bias” was coined by Ni and Zhou. Thus, Torbeyns, Schneider, Xin, and Siegler (2015) are right in arguing that natural number knowledge interference is one, but not the only source of difficulty in rational number learning – and let me add that the brief summary above indicates that the picture is even more complicated than depicted in their article. There is no doubt, however, that natural number knowledge interference is one major source of conceptual difficulties.

3. Theoretical framing of the contributions

The idea of a whole or natural number bias (hereafter, natural number bias) is closely related to the problem of restructuring a prior knowledge base that cannot adequately support a new, and in many ways incompatible, number perspective (Ni & Zhou, 2005; Vamvakoussi, Van Dooren, & Verschaffel, 2012); it is thus related to conceptual change perspectives on the development of rational number knowledge. With the exception of Torbeyns et al. (2015), all contributions are framed in conceptual change terms, focusing on the differences between natural and rational numbers, and studying the complex interactions between students' prior knowledge and the information about rational numbers coming from instruction. Since my view on the matter is expressed in detail elsewhere (e.g., Vamvakoussi & Vosniadou, 2010), and since conceptual change perspectives are also addressed by the contributors, I will focus on the theoretical position of Torbeyns et al. who advocate a different, albeit not incompatible idea. Specifically, Torbeyns et al. focus on the similarities, rather than the differences, between natural and rational numbers. Adopting the integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011), they in fact address the question “what makes natural and non-natural numbers members of the same category, the category of number?” Their answer is “magnitude”. Thus numerical development is described as a process of progressively broadening the class of numbers that are understood to possess magnitudes, are subject to ordering, and can be assigned specific locations on number lines. Siegler et al. made two assumptions: a) fractions are crucial for overall mathematical understanding, and b) understanding magnitudes is crucial for understanding fractions, which is also tested-and supported-by Torbeyns et al. in the present cross-cultural study.

Let me start by saying that I find the idea of exploiting the deep similarities between natural and rational numbers valuable, particularly in terms of educational implications (Vamvakoussi et al., 2013). In fact, this idea has been systematically explored by several mathematics education researchers (e.g., Behr, Harel, Post, & Lesh, 1994; Sophian, 2004; Steffe & Olive, 2010). It is also hard to disagree that magnitude is an essential part of the meaning of number in an abstract context. Yet there is another, possibly more fundamental, commonality, between natural and rational numbers, namely the notion of the unit. The use of the unit differentiates between judgments regarding unquantified and quantified quantities and is thus instrumental for the development of number concepts (Sophian, 2008). The ability to choose or construct appropriate units is considered fundamental for the development of rational number concepts (Harel & Confrey, 1994), notably for the understanding of fraction as measure. And although this fact is often overlooked, the notion of the unit is also essential for natural numbers as well (e.g., Sophian, 2004).

Number magnitude depends crucially on the unit. I believe that this fact is reflected in the findings of Torbeyns et al. (2015). Indeed, one might ask, why is fraction magnitude estimation on the number line (particularly on the 0–5 number line) more difficult than fraction comparisons, and a better predictor of overall mathematical achievement? I would argue that this is because it requires understanding of the role of the unit. This is essential in concrete contexts, such as measurement (Nunes & Bryant, 1996); and more so in abstract ones, such as placing a fraction on the number line. I would not be surprised if several students in the sample of Torbeyns et al. treated the length corresponding to 5 as the unit (see also Ni, 2000).

This said, it is valuable to have research-based evidence that fraction understanding correlates with overall mathematical achievement. One possible explanation of this finding is the pervasiveness of rational number ideas in the curriculum. Rational numbers are an important part of what Vergnaud (1994) termed “multiplicative conceptual field” that includes notions ranging from basic ones such as multiplication and division, to highly sophisticated ones such as n-linear functions. The elements of the multiplicative conceptual field are interrelated, and there is wide variety of mathematical concepts that relate to this field within as well as outside school settings. For instance, proportionality, geometrical similarity, and probability all pertain to the multiplicative conceptual field (see Lamon, 2006, for a detailed discussion). Acknowledging the interconnections between rational number ideas and a wide variety of mathematical notions, many of which are taught at school, one can expect that fraction understanding is important for students' mathematical achievement in different countries (Torbeyns et al., 2015) – and arguably for students' achievement in physics and chemistry, as well.

I would agree with Torbeyns et al. (2015) that conceptual change perspectives focus on a particular (and indeed limited) aspect of the development of rational number knowledge. However, such perspectives (notably, the framework developed by Vosniadou and colleagues, and presented by DeWolf & Vosniadou, 2015) provide detailed accounts of the basis of the natural number bias; predictions of what happens when prior knowledge interacts with new information-and these predictions become more refined (e.g., Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015), as there is now a substantial body of prior research; descriptions and explanations of students' conceptions and how these change (or do not change) with instruction (e.g., Kelley & Rittle-Johnson, 2015; McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015). In fact, intense research from this perspective has put forward several aspects of the natural number bias, which I will discuss in the following section.

The integrated theory, on the other hand, has a more ambitious goal, is a more recent attempt, and thus has more questions to address, in particular *how* do students come to conceptualize natural and non-natural numbers as abstract entities with magnitudes that can be ordered and placed on number lines? At the moment, the integrated theory does not account for the process of development of this understanding. By the fact that the constraining role of natural number knowledge is also acknowledged, I take it that the "progressive broadening" of the category of number is not deemed linear and smooth (consistently with conceptual change perspectives). But if continuity in the shift from natural to rational numbers is also to be assumed, then the specific aspects of natural number knowledge that serve as stepping stones should be accounted for. For instance, does the notion of the unit indeed play a crucial role in this process, as argued above?

From an educational point of view, the integrated theory invests on the aspect of fraction as measure. I strongly agree with this idea (Vamvakoussi & Vosniadou, 2012). However, I would also like to note that this is not the only possible trajectory to fraction learning. Extensive empirical research is required in order to evaluate this proposal, also against alternative proposals (e.g., Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Moss, 2005), especially when it comes to the introduction of fraction ideas to young children.

Either from a basic research perspective, or in terms of educational implications, it appears that the integrated theory (Siegler et al., 2011; Torbeyns et al., 2015) has the potential to develop an interesting and fruitful research agenda.

4. Five aspects of the natural number bias

4.1. Filter of incoming information

This aspect was, in fact, a defining feature of the bias, according to Ni and Zhou (2005): "The whole number bias thus refers to a robust tendency to use the single-unit counting scheme to interpret instructional data on fractions" (p. 28). This definition is most relevant to the framework theory approach to conceptual change-presented in detail by DeWolf and Vosniadou (2015)-since it assumes a knowledge structure that is used as a basis for interpreting information coming from instruction. Because prior knowledge and new information are not compatible, new information on rational numbers may be neglected or distorted. Distortions are the cause of synthetic conceptions (Vamvakoussi & Vosniadou, 2010), such as that "the smaller the components, the bigger the fraction" (DeWolf & Vosniadou, 2015).

4.2. Source of systematic errors

This is probably the most typical aspect of the bias: For every difference between natural and rational numbers, there is a source of systematic errors, due to the fact that students draw on their natural number knowledge to deal with the tasks at hand. This aspect of the bias is, I believe, amply covered by all contributors.

4.3. Of intuitive character

Researchers in the area of conceptual change in general, and of conceptual change in the shift from natural to rational numbers in particular, have noticed that more often than not, students feel overconfident or experience an illusion of understanding precisely when they are heavily biased (Merenluoto & Lehtinen, 2004); and it is precisely the misconceptions that are accompanied with a feeling of certainty that are the most resistant to instruction (Kelley & Rittle-Johnson, 2015). In my research, I have sometimes encountered the "wait a minute!" reaction by students who provided a "biased" answer first, and then revised it themselves. In other cases, students remain insensitive to hints that are intended to make them reconsider their incorrect answers. Consider, for example, one of the experiments reported in Christou and Vosniadou (2012) who were the first to investigate this bias in students' dealing with algebraic expressions: Eight tenth graders (older than the interviewees of the study by Van Hoof et al., 2015) were asked to judge the validity of a series of inequalities. These students either substituted the variables with natural numbers only, or referred to natural number properties, similarly to Van Hoof et al.'s participants; they did not question the correctness of their responses, even after two hints by the interviewer. Finally, they were explicitly asked whether it would be possible to substitute the variables with non-natural numbers; they immediately recognized this as a possibility, and they even seemed a bit surprised that this idea hadn't occurred to them before.

Such features of the bias are captured by Fischbein's (1987) account of intuitive knowledge in science and mathematics. For Fischbein, intuitions are based on complex knowledge structures, formed by in and out-of school experience; they serve as basis for inferences that go beyond the information at hand; and they allow for an immediate and integrated grasp of a situation. Intuitive judgments are fast and appear to be self evident, without the need for further justification. Finally, once established, intuitions are robust and not easily eradicated by instruction – and sometimes co-exist with mathematically correct ideas throughout a person's life. As argued elsewhere (Vamvakoussi et al., 2013), there are several similarities between the framework theory approach to conceptual change (DeWolf & Vosniadou, 2015) and Fischbein's account of intuitions.

Recently, it has been conjectured that the natural number bias has the features of intuitive knowledge. Reaction time studies, based on the distinction between intuitive and analytical reasoning in the frame of dual-process theories, can capture two features of the intuitive character of the natural number bias, namely immediacy and perseverance, particularly in adult populations (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi et al., 2012; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). The idea is that natural numbers and their properties "come to mind first", in the incongruent and also in the incongruent cases, for natural as well as rational number tasks.[2] Thus intuitive judgments emerge first, and lead to correct responses in congruent tasks. In incongruent tasks, however, they either lead to incorrect responses, or to correct responses that take longer, because analytical reasoning is required for the initial response to be inhibited.

DeWolf and Vosniadou (2015) argue that this perspective on the bias is not theoretically sound. I would certainly agree with DeWolf and Vosniadou that intuitive reasoning is not the *source* of the natural number bias. I would rather argue that precisely because of the complex developmental-learning process that DeWolf and Vosniadou refer to, natural number knowledge acquires the intuitive character described above. Because human reasoning is often intuitive in nature, the bias will occasionally manifest itself, sometimes in an error, sometimes in a time-consuming attempt to inhibit the initial intuitive response, and sometimes as illusion of understanding, depending on the task at hand, the individual's familiarity with the task, and their level of engagement.

[2] I feel this clarification is necessary, because DeWolf and Vosniadou (2015) seem to suggest that the dual-process perspective on reasoning is consistent with the idea that intuitive reasoning is applied in natural number tasks, whereas analytical reasoning operates in rational number tasks.

4.4. Distractor in the mental representation of fraction magnitude

More recently, the natural number bias has been linked to fraction magnitude representation. Specifically, componential processing of fractions, mainly in comparison tasks, was deemed a manifestation of the bias. Bonato, Fabbri, Umiltà, and Zorzi (2007) were the first to put forward this claim. They used comparison tasks targeted on unit fractions with single-digit denominators and found that adults processed these fractions componentially, that is, they assessed the magnitude of the denominators and not of the whole fraction (tested using the distance effect and the SNARC effect paradigm). They concluded that "skilled participants (…) do not *automatically* [emphasis added] access the real number that the fraction represents" (p. 1417), even for the simplest of fractions, that is, the inverses of single-digit natural numbers. Now why would this be a manifestation of the natural number bias? From this perspective, a person who answers that "1/6 is bigger than 1/5, because 6 is bigger than 5" and a person who answers that "1/5 is bigger than 1/6, because 5 is smaller than 6" are equally biased. According to Bonato and colleagues, this indicates that educated adults still rely "on a system based on exact, integer numbers even for a stimulus that is intrinsically non-integer" (p. 1411).

Note that Bonato et al. attempted to challenge Gallistel and Gelman's (2000) claim that there is a unique primitive, non-linguistic, system of quantity representation that works with continuous quantities and represents quantitative information about discrete and as well as continuous quantities. It should be stressed that this primitive system is not assumed to represent initially numbers in their symbolic form. An interesting question then is, does it serve as basis for further numerical development? Consistently with the hypothesis of Gallistel and Gelman, evidence coming from numerous studies (see Schneider & Siegler, 2010, for a synopsis) indicates that an analog mental representation, akin to a number line, is used for the representation of natural numbers (also in their symbolic form). Given that rational (and also real) numbers could be, in principle, represented by such a mental representation, Bonato et al., similarly to DeWolf and Vosniadou (2015), reasoned that this would imply that accessing fraction magnitudes should be easy and direct, explicitly using the word automatic. Automaticity is, arguably, a very strong constraint (and is not adequately addressed by Bonato et al.).

At this point, it is worth using a distinction, made by Tzelgov and colleagues (e.g., Kallai & Tzelgov, 2009; 2011; Tzelgov, Ganor-Stern, Kallai, & Pinhas, 2013). These researchers distinguish between primitive and non-primitive representations of number. The term primitive refers to numbers whose magnitude is assumed to be stored in long-term memory and thus can be holistically retrieved in one step; non-primitive, one the other hand, refers to numbers whose magnitude is mentally computed or estimated via the use of their components, rather than accessed directly. According to Tzelgov and colleagues, primitives are empirically identified on the basis of the distinction between automatic and intentional processing, where automaticity is defined as processing without conscious monitoring; and tested mainly within STROOP-like research paradigms (e.g., using the size congruency effect). With this very clear-and quite strict view-on automaticity, Tzelgov and colleagues conducted a series of studies. Their findings indicate that only the mental representations of single-digit natural numbers can be considered as primitives; two-digit natural numbers, decimals, and negative integers require componential processing (Tzelgov et al., 2013). As far as fractions are concerned, it appears that the magnitudes of individual fractions are not automatically retrieved from long-term memory; however, there appears to exist a primitive representation of fraction as an entity "smaller than one" (Kallai & Tzelgov, 2009) – quite consistent with the part-whole aspect of fraction, and the corresponding misconception of many students.

With this distinction in mind, it appears that Bonato et al.'s (2007) study addresses the issue of direct, automatic accessing of fraction magnitudes; their results indicate that even the simplest fractions are not spontaneously processed holistically, a finding consistent with the findings by Kallai and Tzelgov (2009). Schneider and Siegler (2010) argued that the result of Bonato et al. is an artifact of their research tasks, and conducted a series of experiments using a wider variety of fraction comparisons. They found that adults were able to represent fractions on the mental number line (based on the distance effect). Schneider and Siegler's study, although framed as a response to Bonato et al., actually addresses a different question, namely *can* adults access the integrated magnitude of fractions? Not unexpectedly, they can (see also Meert, Grégoire, & Noël, 2010a, 2010b, for 10- and 12-year olds; DeWolf & Vosniadou, 2015). An extensive literature on comparison strategies provides information as to how people do it (I am skipping strategies that circumvent the problem of estimating the magnitude, such as cross-multiplying of the components): benchmarking to 1/2 and 1, or other familiar numbers; residual thinking (i.e., comparing the complementary fractions); estimating the ratio between nominator and denominator; converting the fraction in its decimal form; combinations of the above and other strategies, that may not be documented. In fact, people who are competent in this domain use strategies that are tailored to the task at hand.[3] The picture then becomes more complicated: Sometimes people will not go in the trouble of compiling the magnitude of the fractions, when they can rely on their components. Sometimes accessing the integrated magnitude of fractions coexists with componential processing; and, interestingly, the magnitude of the components, in particular of the denominators, is found to interfere with the process of accessing the integrated magnitude (tested by Meert et al., 2010b, using a priming effect paradigm).

So, is componential processing a manifestation of the whole number bias? In a sense, it is. Consider a special case, unit fractions with single-digit denominators: If the magnitudes of the denominators are indeed automatically activated (Tzelgov et al., 2013), then the individual cannot help the interference of the components in the process of accessing the fraction magnitude, resulting to slower response times, and possibly more errors. This effect might be then intrinsically tied to the way that fractions are represented symbolically, namely as two natural numbers, separated by a bar; and it might have a more drastic constraining effect, than delaying fraction comparison tasks. Consider the study by Kallai and Tzelgov (2011): Educated adults learnt arbitrary, unitary symbols for unit fractions from 1/2 to 1/8, after intensive 4-h training; then, the mental representation of fraction magnitudes were accessed automatically when presented with the arbitrary symbols, but not with the usual symbols. As Kallai and Tzelgov concluded, although the integrated magnitude of specific fractions can be represented holistically in long term memory, their natural

[3] This makes the design of research instruments rather challenging: For instance, the use of 1/2 and 1 as benchmarks was not expected by Schneider and Siegler (2010) and Obersteiner et al. (2013). DeWolf and Vosniadou (2015) have the unexpected result that Greek participants performed better and faster in the inconsistent condition. Taking a closer look at the tasks, one might notice that many inconsistent ones (but not so for congruent ones) have the following characteristic: The distance between the nominators is "small" (in fact, equal to 1 in more than half of the tasks), whereas the difference between the denominators is considerably "larger" (for example, 12/29 vs. 11/18). My strategy for such pairs is based on the idea that I'm getting more or less the same number of "pieces", but the "pieces" are much smaller in the first case (12/29). Of course, this strategy does not work in the general case, but in the inconsistent tasks in question, it always does; and it allows for faster, and accurate, processing of the inconsistent tasks.

number components "interfere with their processing and in fact dominate automatic processing" (p. 9).

If this is the case, then componential processing could be considered as yet another manifestation of the natural number bias, albeit in a slightly different sense than Bonato et al. (2007), and also DeWolf and Vosniadou (2015) intended: The very fact that the symbolic representation of fractions depends on the symbolic representation of natural numbers prevents the magnitudes of the simplest unit fractions from becoming primitives, like one-digit natural numbers.

Would this aspect of the bias then be evidence against the claim put forward by Gallistel and Gelman (2000), as Bonato et al. (2007) and DeWolf and Vosniadou (2015) argue? Before this question can be answered, background assumptions about what automaticity is have to be explicitly stated; and specific predictions about the kind and the range of numbers that could be automatically processed should be formulated. To my understanding, evidence that only the magnitudes of single-digit numbers are primitives (Tzelgov et al., 2013) is more threatening to Gallistel and Gelman's hypothesis, since it is more consistent with the hypothesis that there are core systems that deal with small numerosities (e.g., Le Corre & Carey, 2007).

4.5. Facilitator

Associating the natural number bias with the adverse effects of prior knowledge on further learning (e.g., systematic errors, illusion of understanding) may overshadow its most fundamental aspect, namely that it acts as a facilitator in a wide variety of in and out of school situations. In this special issue this aspect is discussed as the tendency for higher accuracy and lower reactions times in tasks that are compatible with natural number reasoning (DeWolf & Vosniadou, 2015; Van Hoof et al., 2015). In a more general fashion, the facilitator aspect illustrates the value of entertaining a bias of this nature: It allows for immediate judgments, provides a sense of coherence, and satisfies "the fundamental need of human beings to avoid uncertainty" (Fischbein, 1987, p. 28).

5. A note regarding instruction

The facilitator aspect of the natural number bias is typically exploited in instruction. This is why fractions are commonly introduced via their part-whole aspect (Moss, 2005); why young students are taught the double-count technique, that is, to count all equal parts in a pre-divided whole and then count the indicated shaded parts (Kieren, 1992); and why French teachers were instructed to present, for instance, 3.25 as 325 with one hundredth as the unit (Brousseau, 2002). Such approaches initially facilitate students' encounter with non-natural numbers, but they clearly "do not mind the gap" between the natural and the rational number perspective. Although familiar to researchers interested in the development, learning, and teaching of rational number concepts, this idea has not yet made its way to the classrooms. Considering the deep similarities together with the important differences between natural and rational numbers is a promising approach in order to narrow the gap between students' conceptions of numbers, and rational numbers. This proposal requires a long-term perspective on the planning of instruction and extensive empirical validation. Importantly, it also requires from teachers to be convinced that the attempt to make rational number "too easy" for students by building on superficial similarities between natural and rational numbers does not pay off in the long-run.

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