ZDM Mathematics Education (2017) 49:497–507
DOI 10.1007/s11858-017-0857-5

ORIGINAL ARTICLE

Using analogies to facilitate conceptual change in mathematics learning

Xenia Vamvakoussi[1]

Accepted: 21 March 2017 / Published online: 27 March 2017

Abstract The problem of adverse effects of prior knowledge in mathematics learning has been amply documented and theorized by mathematics educators as well as cognitive/developmental psychologists. This problem emerges when students' prior knowledge about a mathematical notion comes in contrast with new information coming from instruction, giving rise to systematic errors. Conceptual change perspectives on mathematics learning suggest that in such cases reorganization of students' prior knowledge is necessary. Analogical reasoning, in particular cross-domain mapping, is considered an important mechanism for conceptual restructuring. However, the use of analogies in instruction is often found ineffective, mainly because the structural similarity between two domains is obscure for students. To deal with this problem, John Clement and his colleagues developed the *bridging strategy* that uses multiple analogies to bring students to pay attention to the structural similarity that often goes unnoticed. This paper focuses on the cross-domain mapping between number and the (geometrical) line that has been instrumental in the development of the number concept. I summarize findings of a series of studies that investigated students' understandings of density in arithmetical and geometrical contexts from a conceptual change perspective; and I discuss how this research-based evidence was used to design an intervention study that used the analogy "numbers are points on the number line", and a bridging analogy ("the number line is like an imaginary rubber band that never breaks, no matter how much it is stressed") with the aim of bringing the notion of density within the grasp of secondary students.

Keywords Conceptual change · Rational numbers · Analogies · Bridging analogies

1 Introduction

The relation between psychology and mathematics education has been long standing. Many psychologists turned to mathematics as an appropriate domain to put theories of learning and development to the test–one can think of Thorndike and Piaget, for example. On the other hand, mathematics education researchers turned to psychology for theoretical accounts of mathematics learning that could, for example, explain students' difficulties in mathematics, describe the processes underlying mathematical reasoning and problem solving, and predict contextual influences on learners' mathematical behaviour (e.g., task features that elicit different responses) (Schoenfeld 1987).

However, the relation between psychology and mathematics education has not been uncontested. For example, it has been argued that psychological theories are misinterpreted and misapplied in the context of mathematics education (see, for example, Anderson et al. 2000). On the other hand, mathematics education researchers came to see the cognitive perspective as too restricted and failing to capture the complex phenomena of learning and teaching mathematics in the classroom; furthermore, the research agenda of the field broadened to include topics such as the socio-political context of learning and teaching mathematics that do not usually pertain to psychological research (Kilpatrick 2014). On top of that, in their attempts to establish

✉ Xenia Vamvakoussi
xvamvak@cc.uoi.gr

[1] Department of Early Childhood Education, University of Ioannina, Building: Metavatiko, University Campus, 45110 Ioannina, Greece

mathematics education as a scientific discipline in its own right, mathematics education researchers set to develop their own theoretical models and research methodologies, turning away from cognitive psychology (De Smedt and Verschaffel 2010). As a result, there is currently great discrepancy between the research foci of psychology and mathematics education even when it comes to mathematics thinking and learning, where larger overlap could be expected (Berch 2016). Recently, however, there appears to be a renewed interest in bridging research on mathematical cognition and development, and mathematics education (e.g., Alcock et al. 2016).

In this article I attempt to illustrate the possibility of productive bridging between psychological and educational research. I will present an overview of a series of studies that investigated students' understanding of the dense ordering (hereafter, density) of rational numbers. These studies were grounded on a conceptual change approach to learning (Vosniadou et al. 2008), taking a cognitive-developmental perspective on the acquisition of rational number knowledge. They resulted in the design of an experimental intervention that systematically employed principles for instruction stemming from conceptual change perspectives on learning, notably the use of bridging analogies and other analogies to foster students' understanding of a highly counter-intuitive idea. I will highlight the relevance of such studies for mathematics education while also discussing their limitations from the point of view of instruction.

2 The problem of conceptual change in the development of rational number knowledge

One of the most well-established findings coming from research on learning in the fields of psychology and education is that prior knowledge plays a very important role in further learning. Attempts have been made to communicate insights about this issue also to educators (Bransford et al. 2000; Donovan and Bransford 2005). Of particular interest are the adverse effects of prior knowledge. These typically occur when the new content to be learnt is incompatible with what the learner already knows. In such cases prior knowledge hinders rather than supports new understandings. An exemplary case is the transition from natural to non-natural numbers. Interference of natural number knowledge in rational number learning has been studied by mathematics educators (e.g., Brousseau 2002; Fischbein 1987; Moss 2005) as well as by cognitive and developmental psychologists (e.g., Hartnett and Gelman 1998; Smith et al. 2005), and has recently attracted the attention of neuro-psychologists as well (e.g., Jacob et al. 2012). This phenomenon is so pervasive that it has been termed whole or natural number bias (Ni and Zhou 2005; Vamvakoussi et al. 2012). The bias manifests itself in various ways, including systematic errors and misconceptions (e.g., *Longer decimals are bigger, multiplication always makes bigger*); faster and more accurate responses to tasks that are compatible with natural number knowledge, but otherwise slower and less accurate responses; and unjustified feelings of certainty when incorrect responses are provided to incompatible tasks (see Vamvakoussi 2015, for a brief overview of the different aspects of the natural number bias).

The transition from natural to rational numbers has been acknowledged as one that requires conceptual change (Ni and Zhou 2005; Smith et al. 2005; Vamvakoussi and Vosniadou 2010; Vosniadou et al. 2008). In the context of educational research, the term *conceptual change* refers to the process of knowledge restructuring that is necessary when one is exposed to information that is not compatible with one's prior knowledge (Schneider et al. 2012). The conceptual change perspective on learning has been prominent in science education research. Science education researchers were interested in how concepts change in the process of learning science under instruction (see the seminal paper by Posner et al. 1982), with a view to account for, and address, the phenomenon of students' systematic misconceptions. In 2004, an attempt was initiated to take a conceptual change perspective on mathematics learning (Verschaffel and Vosniadou 2004). A specific approach to conceptual change, namely the framework theory approach (Vosniadou et al. 2008), originally developed to explain students' conceptual difficulties in science learning, has since been fruitfully applied in the area of mathematics learning (see Vamvakoussi et al. 2013, for an overview of related research). A key assumption of this approach is that from early on children organize their interpretations of everyday experiences into few, relatively coherent conceptual structures termed *framework theories*. Framework theories allow children to generate explanations for various phenomena, make predictions, and deal with unfamiliar situations. One such framework theory pertains to the domain of number. Indeed, evidence stemming from cognitive-developmental research indicates that, in the context of lay culture, preschoolers have already formed a principled understanding of numbers as *counting numbers*, which is further enriched and strengthened by early instruction that focuses on natural numbers (Gelman 1990, 2000; Ni and Zhou 2005; Smith et al. 2005). Thus, before being exposed to rational number instruction, students have constructed rather structured framework theories of number which are based on informal and formal experiences of natural number features and properties, and shape their beliefs and expectations about what numbers are and how they function (see Vamvakoussi and Vosniadou 2010, for a detailed account of students' framework theories of number). When non-natural numbers are introduced in the curriculum, practically every background assumption of these theories is no longer valid. In their attempts to make sense of non-natural numbers, students draw heavily on their prior knowledge about natural numbers, which results in systematic errors precisely in tasks that touch upon the differences between the natural and the rational numbers.

Restructuring the initial framework theories of number is a challenging and time-consuming process, because it involves the revision of a system of interrelated ideas about numbers that requires (among others) re-evaluation of deeply entrenched assumptions (e.g., numbers are discrete); ontological shifts (e.g., natural and non-natural numbers are members of the same category); representational changes (e.g., from the number sequence, to dense intervals); and re-evaluation of numerous procedures and strategies that do not apply to rational numbers (e.g., the value of a decimal number cannot be judged by the number of its digits).

Thus the problem of conceptual change in the transition from natural to rational numbers presents mathematics educators with a big challenge. The question arises whether principles for instruction stemming from conceptual change perspectives on learning can be useful for instruction aiming at addressing this major source of difficulties for students.

Conceptual change perspectives on learning have been initially associated with cognitive conflict as a main instructional strategy. The central component of this strategy is to confront students with information that contradicts their current ideas. Although sometimes effective, this strategy has been criticized on several grounds (Limón 2001; Merenluoto and Lehtinen 2004; Smith et al. 1993). For example, it has been pointed out that what constitutes a conflict from the point of view of the teacher is not necessarily perceived as such by the students, because they may neglect or misinterpret the contradictory information (when, for example, they are not fully aware of their own ideas or when they are overconfident in them); that it is not easy for all students to handle cognitive conflict productively, because it requires substantial effort that a student might not be motivated to put into this task; and that inducing cognitive conflict may create feelings of uncertainty, or even frustration, that some students find difficult to handle. It was thus acknowledged that cognitive conflict should be used with caution. Moreover, it was also realized that teaching for conceptual change is a complex enterprise that cannot rely solely on cognitive conflict. Several principles for instruction aiming at conceptual change have been put forward and tested (e.g., Vosniadou et al. 2001). This paper focuses on the use of analogies in instruction aiming at conceptual change, in particular on the bridging analogy approach. This approach has been proposed as a strategy that fosters conceptual change by building on students' productive ideas instead of emphasizing their misconceptions, and has been implemented in the context of science education with positive outcomes (Clement 2008).

In the following the theoretical underpinnings for the value of analogical reasoning in conceptual restructuring as well as its relevance for mathematics learning will be discussed, with focus on the development of the number concept.

2.1 Analogical reasoning and conceptual change

Analogical reasoning relies on the comparison between two systems that can belong to the same, to similar or to different domains, and are perceived as similar in some respects. The one deemed more familiar to the individual is termed "the source", while the less familiar one is termed "the target". Analogical reasoning involves mapping between the source and the target, that is, finding correspondences between the two systems. If a solid match is accomplished, based on structural rather than superficial similarities, then knowledge of the source can be employed productively to draw inferences about the target. Analogical reasoning is acknowledged by psychologists as well as mathematics educators as a process than can trigger the generation of hypotheses about an unfamiliar situation, serve as a source of problem-solving strategies and, more generally, as an aid to discovery and learning. (e.g., English 1997; Gentner et al. 1997).

Cross-domain mapping is considered a "bootstrapping process" that supports learning when what is to be learned transcends what is already known not merely in a quantitative, but also in some qualitative way (Carey 2004; Smith et al. 2005); and, importantly, as a mechanism for conceptual restructuring, since in the process of comparing the source and the target, either or both domains may be re-organized to improve the match (Gentner et al. 1997; Gentner and Wolff 2000; Vosniadou 1989).

Cross-domain mapping, especially between physical quantity and number, is an extremely important mechanism for the development and restructuring of number concepts in the individual (Resnick and Singer 1993; Siegler 2016; Smith et al. 2005). For example, Smith et al. (2005) argue that cross-domain mapping between physical quantity and number can bring children to see numbers as infinitely divisible.

Cross-domain mapping has been also been instrumental for the development and restructuring of mathematical knowledge in the context of discovery. Taking a strong position, Núñez and Lakoff (2005) argue that practically all mathematical ideas rely on cross-domain mappings from one conceptual domain to another which they call *conceptual metaphors*.[1] A prominent example of cross-domain mapping in the history of mathematics is the complex interplay between arithmetic and geometry (Dantzig 2005). The central concept of arithmetic, namely number, and the straight line, started as two radically different objects of study (numbers were deemed discrete, while the line was deemed the exemplar of continuity), and ended up related by the numbers-to-points correspondence. The comparison between number and the line, acting interchangeably as source and target domains has been associated with the development of powerful mathematical ideas and tools, such as analytic geometry and the calculus, and was instrumental in the development and formalization of the concepts of number, infinity, and continuum. In this process, the concept of number as well as of the line underwent radical changes of meaning: The category of number, initially including only the natural numbers, changed to encompass non-natural numbers; and the straight line was reconceptualised as set of points (Lakoff and Núñez 2000; see also; Vamvakoussi and Vosniadou 2012).

Analogies that rely on the comparison between two different domains are pervasive in mathematics instruction. A prominent example is the use of concrete or graphic/diagrammatic representations of abstract mathematical ideas (English 1997). Consider, for example, that the main external representations for natural numbers in the early years build on the mapping between physical (discrete) quantity and number, via the act of counting. Consider also that the analogy "numbers are points on the line", a product of the long-term comparison between the domains of arithmetic and geometry, underlies an extensively used representation of numbers in today's mathematics classrooms, namely the number line.

With this last example in mind, it can be noted that analogies used (explicitly or implicitly) in mathematics instruction can be of a very abstract nature, rich in very elaborate meanings, and thus not readily accessible to students. Indeed, despite the fact that analogical reasoning is a fundamental aspect of human cognition that is employed spontaneously even by young children to make sense of unfamiliar phenomena, it has been amply documented that when people are presented with specific analogies and asked to use them, they fail to a considerable extent (Duit 1991; Dunbar 2001). This phenomenon becomes a problem when it comes to analogies used in instruction. There are several reasons why the use of analogies in instruction does not always yield the expected results (Clement 2008): Sometimes the analogies used are simply not appropriate. Other times, however, the source is not as familiar to students as expected; or students may interpret it on the basis of their initial ideas that are not consistent with the intended scientific ideas. Often, the relational similarities between the source and the target are not transparent to students. It thus appears that extensive support is required in instruction for students to be able to use the intended analogy productively (Richland et al. 2007).

Clement and colleagues (e.g., Brown and Clement 1989; Clement 1993, 2008) proposed the *bridging analogies* teaching strategy that involves the purposeful interpolation between students' initial understanding of a situation and the intended scientific idea of one or more intermediate anchoring situations. An anchoring situation is close to students' initial ideas, but is also compatible with the intended scientific idea; it is thus expected to trigger a favourable intuition, that is, an intuition that can be developed toward the understanding of the target situation. For example, in a situation where a book lies on a table, students typically do not see the table as exerting any force on the book, which is a common misconception about acting-reacting forces. The researchers designed interventions based on an anchoring example (i.e., a book on a spring) that was followed by bridging examples (e.g., a book on foam rubber) until they gradually reached the target situation (i.e., the book on the table).

The bridging analogies strategy has been extensively employed in science education research by Clement and colleagues, but also by other research groups (e.g., Bryce and Macmillan 2005; Savinainen et al. 2005; Yilmaz et al. 2006), and has proved fairly successful in bringing students to reconsider and revise their initial ideas about scientific phenomena that were inconsistent with the scientific view.

In the following I illustrate how an analogy ("numbers are points on the line") together with a bridging analogy ("the number line as a rubber line") was used in the design of an experimental intervention aiming at fostering students' understanding of the dense ordering of rational numbers.

2.2 Tapping into students' understanding of the density property of rational numbers

Unlike natural numbers, rational numbers are densely ordered. There are different ways to describe what dense order is, all of which are equivalent from a mathematical point of view: (a) between any two, non-equal, rational numbers there is always another rational number (and, thus, there are infinitely many intermediates), (b) no rational number has a (unique) successor in the rational numbers

[1] There is a debate regarding the relation between analogy and metaphor. It is beyond the scope of this paper to enter this discussion. Following Bowdle and Gentner (2005), we take (conceptual) metaphor to be a species of analogy.

set, (c) there is no least positive rational number, and (d) rational numbers are infinitely divisible.

Historically, the idea of density emerged in a geometrical context, as a defining property of continuous quantities, notably of the straight line. It took centuries before it was transferred to the domain of number, and differentiated from continuity (see Vamvakoussi and Vosniadou 2012, for a more detailed account). Interestingly, the cross-domain mapping between number and the line was, as already mentioned, instrumental in this respect.

In principle, the density of rational numbers is accessible via simple procedures that are taught at the elementary school. For example, one can always find (some more) fractions between two given fractions (e.g., 2/5 and 3/5) by converting them to equivalent forms (e.g., 4/10 and 6/10); one can always find (some more) decimals between two decimals (e.g., 0.5 and 0.6) also by converting them to equivalent forms (e.g., 0.50 and 0.60); and this process can be repeated again and again. However, the density property is notoriously difficult for students from elementary and secondary (McMullen et al. 2015; Merenluoto and Lehtinen 2002), up to tertiary education (Giannakoulias et al. 2007). A repeating finding is that students typically assign the property of discreteness to rational numbers, which is interpreted as an intrusion of natural number knowledge. In the following I summarize a series of studies that looked into students' understandings of density from a conceptual change perspective and provided a detailed picture of students' difficulties.

In a series of studies (Vamvakoussi et al. 2003, 2011; Vamvakoussi and Vosniadou 2004, 2007, 2010, 2012) we investigated secondary students' (7th to 11th graders) understanding of the density of rational numbers as well as of the points on a straight line segment. A variety of tasks (see also Van Dooren et al. 2013) was designed to tap into students' understandings, including open and forced-choice items; construction of models for numbers, for the number line and for the geometrical line; comparison between students' models and models used in instruction; and thought experiments (e.g., *Imagine that you can become as small as a point of the number line. Then you could see the other points up close. Suppose that you are on the point that stands for the number 2.3. Can you define what point is the one closest to you? Describe in words or by drawing a picture*). These tasks targeted mainly the "infinitely many intermediates" and the "no successor" aspect of density and were used in qualitative as well as quantitative studies, including intervention studies.

As expected, we found clear interference of natural number knowledge in the arithmetical context. Specifically, students responded very frequently that there is a finite number (often, zero) of intermediates between two rational numbers. In addition, we found that the kind (natural/non-natural number) and the representational form (fraction/decimal) of the interval end points had a strong effect on students' judgments about the number of intermediate numbers. More specifically, students were more prone to accept the infinity of intermediates between two natural numbers than between fractions or decimals; they might answer that there are infinitely many numbers between two decimals, but a finite number of intermediates between two fractions, or vice versa; they might answer that there are infinitely many intermediates, but of the same representational form as the interval end points (i.e., infinitely many decimals between decimals, and infinitely many fractions between fractions). These findings suggested that students' conceptualizations of rational numbers were far from the view of the rational numbers set as a unified system of numbers that are invariant under different symbolic representational forms. This interpretation was further corroborated by students' own models as well as verbal descriptions of the rational numbers set.

On the other hand, we found that students performed better in a geometrical context (Vamvakoussi and Vosniadou 2012). Indeed, students were more likely to answer that there are infinitely many points between two points of a straight line, than to answer that there are infinitely many numbers between two numbers. This finding should not be taken to suggest that students had a firm understanding of the infinity of points of a line segment. For instance, we also found that students were susceptible to variations of the segment length ("more points on a longer segment"). In addition, the great majority of students described the segment as a "necklace of beads" (Sbaragli 2006) or as a continuous albeit two dimensional object that gets larger in width when it gets magnified. These findings are consistent with the assumption that students do not differentiate between the idealized geometrical objects and their physical representations, a phenomenon that has been noted in the literature (e.g., Fischbein 1987). For some students it is also possible that when referring to the infinity of points on a line segment, they actually refer to a very large number of very small entities, like grains of a very fine powder.

What is interesting is that whatever understanding students had of the infinity of points in a line segment did not transfer to numbers, even when the numbers were presented on the number line (Vamvakoussi and Vosniadou 2007, 2012). In fact, sometimes the presence of the number line had a negative effect on students' responses, causing them to move from "infinitely many intermediates" in the numerical context to "a finite number of intermediates" in the presence of the number line. Consider, however, that students are from early on exposed in instruction to experiences with the "number track" (i.e., a sequence of numbers typically placed in a series of adjacent squares) and the ruler (Doritou and Gray 2009; English 1997), which may explain this finding.

Finally, we found that from the students' point of view, the infinity of intermediates was not equivalent to the "no successor" principle, neither in arithmetical nor in geometrical context: Students who were on the "infinity of intermediates" side could nevertheless believe that there is a successor to a given number or point. One could argue that this finding again reflects students' interpretation of the expression "infinitely many" as "a very large amount". However, we also had evidence coming from in-depth interviews showing that students who actually were able to describe a mechanism producing infinitely many numbers in an interval, and who even stated that the successor of a given number cannot be pinned down precisely *because of the infinity of numbers*, still insisted that there is a successor (Vamvakoussi 2010). This finding becomes less surprising, if one considers how students reach the conclusion that there are infinitely many numbers in a given interval: Students typically rely on recursive processes, very similar to the ones described above, that consist of discrete steps, each step producing a finite number of intermediates. In our studies we had the opportunity to witness some students reach this conclusion on the spot, presumably because they had never been asked this question before. When asked, for instance, about the intermediates of 0.2 and 0.3, some students typically started by presenting 0.21, 0.22, 0.23, and so on, as examples. Then they thought of the possibility of adding another decimal digit, and came up with 0.211, 0.212 and so on. At some point, they realized that they can always add another decimal digit. They concluded that there are infinitely many intermediates, in the sense that there are always more to be found, by adding more decimal digits. This realization, however, does not necessarily imply the representation of these numbers as a dense array.

2.3 Using bridging analogies and other analogies to foster students' understanding of density

Based on our insights about students' prior understandings of density, we looked for the kind of intervention that could bring within the grasp of students this notion, particularly its "no successor" aspect that appeared to be the most challenging one. We hypothesized that an intervention building on the cross-domain mapping between number and the line, via the "numbers are points on the line" analogy, could be effective in this respect. The geometrical line was selected as source, since it appeared that the idea of density was more accessible to students in this context. In addition, it would allow us to circumvent the problem of interpretation of rational number notation, which was an additional challenge for our participants. In fact, the analogy "numbers are points on the line" has the potential to support students' understanding of rational numbers as individual entities, invariant under different forms of representation and, eventually, the rational numbers as a unified system of numbers (Kilpatrick et al. 2001). However, it should be evident from the discussion above that there was a considerable gap between students' interpretations of the geometrical line, and the sophisticated idea of the line as a dense array of points that are not arranged such that one is immediately next to the other. Inspired by the bridging analogies approach of Clement and colleagues, we devised the "rubber line" (i.e., an imaginary elastic line that never breaks, no matter how much it is stretched) as a bridging device. We reasoned that "the rubber line" could be effective because it evokes students' experiences with a real world object, that is, the rubber band; it is consistent with students' experiences with physical representations of the number line; it is associated, via the imaginary property of being unbreakable, with a recursive process, which is an easier way of accessing the notion of infinity for students; and, finally, this process produces a sequence of segments, rather than a sequence of points that can be deemed discrete. We hypothesized that the "rubber line" has the potential to help students grasp the idea that points can never be found such that one lies next to the other.

This hypothesis was tested in a short, text-based intervention (Vamvakoussi and Vosniadou 2012). We produced a text that provided explicit information about the infinity of numbers in a specific interval; made explicit reference to the numbers-to-points correspondence; and employed the analogy of the rubber line to convey the idea that points (and numbers) can never be found one immediately next the other. The excerpt regarding the bridging analogy reads as follows:

> *How is it possible for all these numbers to be placed in the interval between 0 and 1 on the number line? Are there enough points available?*
> *The mathematical number line is a strange object. You can imagine it as a rubber band that never breaks, no matter how much you stretch it. Place numbers between 0 and 1, until it looks like you have used all the available points. If you stretch the rubber band, then you will find out that between the points that looked as if there were the one next to the other, there are more available points, corresponding to more numbers. This procedure can be repeated infinitely many times—don't forget that your imaginary rubber band never breaks! So, there are infinitely many points between the points corresponding to 0 and 1—therefore, there are infinitely many numbers between 0 and 1.* (p. 284).

We tested experimentally the value of the "rubber line" comparing it to two other texts that presented the same

➤ **Kyriakos says:** *The number 2.38 is the successor of 2.37 because 38 is the successor of 37.*

➤ **Do you agree with Kyriakos? Yes ❑ No ❑**

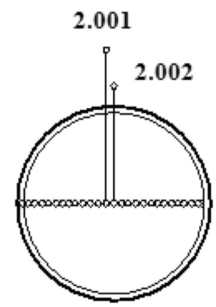
➤ **Explain your answer:**

➤ **Anna says:** *The successor of 3 is the number 3.0000…01. The three dots mean that there are many-many zeros.*

➤ **Do you agree with Anna? Yes ❑ No ❑**

➤ **Explain your answer:**

➤ **Kostas says:** *The numbers 2.001 and 2.002 correspond to two successive points on the number line. If I had an imaginary microscope so I could see these points and their nearby points from really close, I would see the following picture:*



➤ **Do you agree with Kostas? Yes ❑ No ❑**

➤ **Explain your answer:**

Fig. 1 Tasks targeting the "no successor" aspect of density

information as the experimental text, except for the rubber line bridging analogy that was replaced by examples of intermediate numbers in one of the texts, and by figures illustrating the examples in the other. Six classes of 8th and 11th graders (140 students in total) received a pre-test with density tasks in arithmetical and geometrical context, targeting the "infinitely many intermediates" aspect of density; then each class received one of the texts (i.e., one class per grade received the experimental text); and finally, they received a post-test containing all the tasks of the pre-test, and 5 additional tasks that examined whether students were able to deal with the "no successor" aspect of density (see Fig. 1).

Our results showed that all groups profited from the explicit information about the infinity of numbers presented in all three texts, improving their performance in the "infinitely many intermediates" items. However, the experimental group outperformed the other groups in the "no successor" items of the post-test. In addition, the students of the experimental group were more consistent in providing correct answers across contexts (i.e., for numbers, the line, and the number line); and more consistent in providing explanations for their answers. Their explanations indicated that they had employed effectively the ideas underlying the rubber line bridging analogy. Consider, for example, the following response (in written text) of a student reacting to the last task presented in Fig. 1, keeping in mind that, based on her responses in the pre-test, this student was clearly on the discreteness side before the intervention.

> *Between 2.001 and 2.002 there are infinitely many points. Actually, if we think of the number line, that we can stretch (mentally) as much as we like, then we understand that it is not possible to make a picture showing infinitely many tiny spots between 2.001 and 2.002.* (Vamvakoussi and Vosniadou 2012, p. 279)

To sum up, we designed this intervention on the basis of an analogy between two different domains, building the intended mathematical idea in the source that we took to be more accessible to students, and using a bridging analogy to decrease the gap between students' initial ideas of the source and the intended ones. It should be stressed that this intervention relied heavily on the insights into students' thinking, gained through our previous studies.

3 Discussion

In this paper I focused on a problem that is relevant to psychology as well as to mathematics education, namely the problem of natural number interference in rational number learning. This problem can be placed in the more general frame of the role of prior knowledge in further learning. I presented the rationale of an experimental intervention that was based on a principle for instruction stemming from conceptual change perspectives on learning, namely the use of analogies and the bridging analogies strategy to foster students' understanding of ideas that are qualitatively different from their current understandings (Clement 2008; Vosniadou et al. 2008).

The study has certain merits: It is theoretically grounded, and built upon careful analysis of the targeted mathematical concept as well as on research-based evidence of students' understandings of density; it involves the development and assessment of a specific instructional tool and it shows that this tool can have a positive impact on students' understanding of a highly counter-intuitive concept, even with a minimal intervention. This said, how relevant is this study from a mathematics education perspective?

It could be argued that the study is narrow in its scope and deals with only a very small fraction of the challenges faced by educators in rational number teaching. Moreover, the targeted concept does not have a central place in the mathematics curriculum–in fact, density is not even explicitly addressed in some curricula, such as the Greek and the Flemish ones (Van Dooren et al. 2013). In addition, the intervention was restricted to imposing text-based information on students and did not allow for teacher-student or student–student interaction in the classroom; clearly, this is not an optimal learning environment from the point of view of instruction. One could also ask, after asserting that conceptual change is a difficult and slow process, is it possible to expect that such a short-term intervention could have a substantial impact on students' conceptualization of numbers, or of the line?

The above concerns are not hypothetical ones. In fact, they were explicitly expressed by the reviewers of the manuscript reporting this particular study, who voiced (some of) the tensions between research that is more cognitively-oriented and research that is more educationally-oriented. Moreover, all concerns are justifiable from the point of view of an educator who seeks to maximize the opportunity for students to learn: Why should one spend valuable teaching time on a highly abstract and counter-intuitive concept, if it's not among the curricular goals? And why should one restrict oneself to using a text? Had this intervention allowed for more interaction in the classroom, for more feedback by the teacher, for the use of dynamic representations via educational software, and so on, it would have arguably been more effective.

It should be acknowledged that such a short-term, experimental intervention cannot be expected to have substantial or sustainable learning outcomes for students. What it can do, however, is to establish that the underlying principles can be of value to education, provided that they are used systematically and on a long-term basis. In the following I will attempt to draw some recommendations, which are compatible with conceptual change perspectives on learning, and could arguably be appreciated by mathematics educators as well. These recommendations are by no means exhaustive, but are related to the series of studies presented above and to the issues discussed.

Keep in mind that even mathematical ideas deemed "simple" are in fact highly complex and abstract Consider, for example, that fractions and decimals are introduced as "numbers" in instruction in a matter-of-fact way, although they differ in many important respects from what students recognize as "numbers"–for instance, they do not answer a "how many?" question. Substantial effort is required for students to acknowledge non-natural numbers as full-fledged numbers, members of the same family as the natural numbers. This requires focusing on the deep similarities between natural and non-natural numbers, e.g., highlighting similarities between counting and measuring, pointing to the fact that natural and non-natural numbers have magnitudes, and that they are all placed on the (same) number line (see also Siegler 2016).

Take students' prior knowledge into account This principle is so widely acknowledged that it may seem trivial to repeat it here. However, there are different ways to take prior knowledge in account in instruction, and it is worth summarizing them.

First, the adverse effects of prior knowledge should be considered. According to Moss (2005), an important problem in rational number instruction is that not enough attention is paid to students' struggle to assign meaning to rational numbers. More opportunities for students to externalize and negotiate their ideas about numbers need to be provided in the classroom, so that they become aware of them and that are able to evaluate and eventually revise them (Vosniadou et al. 2001). It could be noted here that the use of not-typical tasks, such as the tasks about density used in our studies, can have a value in this respect.

Second, the positive role of prior knowledge should be considered. In fact, conceptual change perspectives on learning have been criticized on the grounds that they overemphasize the adverse effects of prior knowledge, neglecting students' productive ideas that can serve as a basis for further learning (Smith et al. 1993). The *bridging strategy* discussed in this article is but one example of constructive use of students' ideas in instruction, aiming at inducing conceptual change. Another elaborate example in the area of fraction learning can be found in the work of Steffe and Olive (2010).

Third, when capitalizing on students' prior knowledge, the long-term consequences should also be taken into consideration. Indeed, not all aspects of prior knowledge are productive in the long run (Resnick 2006). A prominent example is the over-emphasis on the part-whole aspect of fraction, represented with the area model, which allows for students to employ their natural number knowledge but creates many difficulties in the long run (Moss 2005).

Pay attention to the analogies used in the classroom The use of analogies is supposed to bring within the grasp of students the essentially abstract mathematical ideas. However, great caution is necessary in the selection of analogies, since it is possible they may stand in the way of further learning. Consider, for example, that students' early encounters with the "number track" might convey the idea that numbers are discrete (English 1997). Anticipating latter expansions of the meaning of the term number, one could consider starting with continuous number lines (that do not start at zero).

Furthermore, one should be aware that analogies are not necessarily transparent to students. Substantial support is required for students to perceive the intended relational similarities and to use them productively (Richland et al. 2007). Using bridging analogies (Clement 2008) is one way to support students that has been validated by science education research. As indicated by our study, this approach could be fruitfully applied also to mathematics learning.

Invest in the cross-domain mapping between numerical and geometrical objects Cross-domain mapping between physical quantity and number is a fundamental process in the development of number concepts (Smith et al. 2005; Moss 2005). Cross-domain mapping between numbers and the line goes beyond comparing concrete and abstract entities, towards a comparison between abstract entities that has been instrumental in the historical development of the number concept. Purposeful, long-term investment in this mapping in instruction might foster students' conceptualization of numbers as individual entities, invariant under different symbolic representations; and non-natural numbers as members of the same category as natural numbers (see also Kilpatrick et al. 2001).

For these recommendations to be truly useful for educators, further research is necessary focusing on the development, assessment, and implementation of curricular materials that build on these ideas. This kind of research lies in the intersection of cognitive-developmental psychology and education and could be an example of a bridge between the two fields.

References

Alcock, L., Ansari, B., Batchelor, S., Bissona, M. J., De Smedt, B., Gilmore C. ,..., & Webe, K (2016). Challenges in mathematical cognition: a collaboratively-derived research agenda. *Journal of Numerical Cognition, 2*, 20–41. doi:10.5964/jnc.v2i1.10.

Anderson, J.R., Reder, L.M., & Simon, H.A. (2000). Applications and misapplications of cognitive psychology to mathematics education. *Texas Educational Review*. Retrieved on 2016 August 15 from http://act-r.psy.cmu.edu/?post_type=publications&p=13741.

Berch, D. B. (2016). Disciplinary differences between cognitive psychology and mathematics education: A developmental disconnection syndrome. *Journal of Numerical Cognition, 2*(1), 42–47. doi:10.5964/jnc.v2i1.23.

Bowdle, B. F., & Gentner, D. (2005). The career of metaphor. *Psychological Review, 112*(1), 193–216. doi:10.1037/0033-295X.112.1.193.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academy Press.

Brousseau, G. (2002). *Theory of didactical situations in mathematics (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. and Trans.)*. New York: Kluwer Academic Publishers.

Brown, D. E., & Clement, J. (1989). Overcoming misconceptions via analogical reasoning: Abstract transfer versus explanatory model construction. *Instructional Science, 18*, 237–261. doi:10.1007/BF00118013.

Bryce, T., & Macmillan, K. (2005). Encouraging conceptual change: the use of bridging analogies in the teaching of action–reaction forces and the 'at rest'condition in physics. *International Journal of Science Education, 27*, 737–763. doi:10.1080/09500690500038132.

Carey, S. (2004). Bootstrapping and the origin of concepts. *Daedalus, 133*, 59–68. doi:10.1162/001152604772746701.

Clement, J. (1993). Using bridging analogies and anchoring intuitions to deal with students'preconceptions in physics. *Journal of Research in Science Teaching, 30*, 1241–1257.

Clement, J. (2008). The role of explanatory models in teaching for conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (1st ed., pp. 417–452). Mahwah, NJ: Lawrence Erlbaum Associates.

Dantzig, T. (2005). *Number: The language of science* (4th edn.). New York: Pi Press.

De Smedt, B., & Verschaffel, L. (2010). Travelling down the road: From cognitive neuroscience to mathematics education …and back. *ZDM—The International Journal on Mathematics Education, 42*, 649–654. doi:10.1007/s11858-010-0282-5.

Donovan, M. S., & Bransford, J. D. (2005). *How students learn: History, mathematics, and science in the classroom*. Washington, DC: The National Academies Press.

Doritou, M., & Gray, E. (2009). Teachers' subject knowledge: the number line representation. *Paper presented at 6th Conference of the European society for Research in Mathematics Education (CERME 6)*, Lyon, France.

Duit, R. (1991). On the role of analogies and metaphors in learning science. *Science Education, 75*, 649–672. doi:10.1002/sce.3730750606.

Dunbar, K. (2001). The analogical paradox: Why analogy is so easy in naturalistic settings yet so difficult in the psychological laboratory. In D. Gentner, K. J. Holyoak & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 313–3340). Cambridge: The MIT Press.

English, L. D. (1997). Analogies, metaphors, and images: Vehicles for mathematical reasoning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 3–18). Mahwah, NJ: Erlbaum.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: Kluwer Academic Publishers.

Gelman, R. (1990). First principles organize attention to and learning about relevant data: Number and animate-inanimate distinction as examples. *Cognitive Science, 14*, 79–106. doi:10.1207/s15516709cog1401_5.

Gelman, R. (2000). The epigenesis of mathematical thinking. *Journal of Applied Developmental Psychology, 21*, 27–37. doi:10.1016/S0193-3973(99)00048-9.

Gentner, D., Brem, S., Ferguson, R. W., Markman, A. B., Levidow, B. B., Wolff, P., & Forbus, K. D. (1997). Analogical reasoning and conceptual change: A case study of Johannes Kepler. *Journal of the Learning Sciences, 6*(1), 3–40. doi:10.1207/s15327809jls0601_2.

Gentner, D., & Wolff, P. (2000). Metaphor and knowledge change. In E. Dietrich & A. Markman (Eds.), *Conceptual change in humans and machines* (pp. 295–342). Mahwah, NJ: Lawrence Erlbaum Associates.

Giannakoulias, E., Souyoul, A., & Zachariades, T. (2007). Students' thinking about fundamental real numbers properties. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 416–425). Cyprus: ERME, Department of Education, University of Cyprus.

Hartnett, P., & Gelman, R. (1998). Early understandings of numbers: Paths or barriers to the construction of new understandings? *Learning and Instruction, 8* ,341–374. doi:10.1016/S0959-4752(97)00026-1.

Jacob, N. J., Vallentin, D., & Nieder, A. (2012). Relating magnitudes: The brain's code for proportions. *Trends in Cognitive Science, 16*, 157–166. doi:10.1016/j.tics.2012.02.002.

Kilpatrick, J. (2014). History of research in mathematics education. In S. Lehrman (Ed.), *Encyclopedia of mathematics education* (pp. 267–271). London: Springer.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding + it up. Helping children learn mathematics*. Washington, DC: National Academy Press.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Limón, M. (2001). On the cognitive conflict as an instructional strategy for conceptual change: A critical appraisal. *Learning and Instruction, 11*, 357–380. doi:10.1016/S0959-4752(00)00037-2.

McMullen, J., Laakkonen, E., Hannula-Sormunen, M., & Lehtinen, E. (2015). Modeling the developmental trajectories of rational number concept (s). *Learning and Instruction, 37*, 14–20. doi:10.1016/j.learninstruc.2013.12.004.

Merenluoto, K., & Lehtinen, E. (2002). Conceptual change in mathematics: Understanding the real numbers. In M. Limon & L. Mason (Eds.), *Reconsidering conceptual change: Issues in theory and practice* (pp. 233–258). Dordrecht: Kluwer.

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Towards a systemic model of the processes of change. *Learning and Instruction, 14*, 519–534. doi:10.1016/j.learninstruc.2004.06.016.

Moss, J. (2005). Pipes, tubes, and beakers: New approaches to teaching the rational number system. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: Mathematics in the classroom* (pp. 121–162). Washington, DC: National Academic Press.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27–52. doi:10.1207/s15326985ep4001_3.

Núñez, R., & Lakoff, G. (2005). The cognitive foundations of mathematics: The role of conceptual metaphor. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 109–124). New York, NY: Psychology Press.

Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of a scientific conception: Towards a theory of conceptual change. *Science Education, 66*, 211–227. doi:10.1002/sce.3730660207.

Resnick, L., & Singer, J. (1993). Protoquantitative origins of ratio reasoning. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107–130). Hillsdale, NJ: Erlbaum.

Resnick, L. B. (2006). The dilemma of mathematical intuition in learning. In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 173–175). Prague: PME.

Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogies in the mathematics classroom. *Science, 316*, 1128–1129. doi:10.1126/science.1142103.

Savinainen, A., Scott, P., & Viiri, J. (2005). Using a bridging representation and social interactions to foster conceptual change: Designing and evaluating an instructional sequence for Newton's third law. *Science Education, 89* (2), 175–195. doi:10.1002/sce.20037.

Sbaragli, S. (2006). Primary school teachers' beliefs and change of beliefs on mathematical infinity. *Mediterranean Journal for Research in Mathematics Education, 5*(2), 49–76.

Schneider, M., Vamvakoussi, X., & Van Dooren, W. (2012). Conceptual change. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning* (pp. 735–738). New York: Springer.

Schoenfeld, A. H. (Ed.). (1987). *Cognitive science and mathematics education*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science, 19*, 341–361. doi:10.1111/desc.12395.

Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology, 51*, 101–140. doi:10.1016/j.cogpsych.2005.03.001.

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences, 3*(2), 115–163. doi:10.1207/s15327809jls0302_1.

Steffe, L. P., & Olive, J. (2010). Children's fractional knowledge. New York, NY: Springer.

Vamvakoussi, X. (2010). The 'numbers are points on the line' analogy: Does it have an instructional value? In L. Verschaffel, E. De Corte, T. de Jong & J. Elen (Eds.), *Use of external representations in reasoning and problem solving: Analysis and improvement*. New Perspectives on Learning and Instruction Series (pp. 209–224). New York, NY: Routlege.

Vamvakoussi, X. (2015). The development of rational number knowledge: Old topic, new insights. *Learning and Instruction, 37*, 50–55. doi:10.1016/j.learninstruc.2015.01.002.

Vamvakoussi, X., Christou, K. P., & Van Dooren, W. (2011). What fills the gap between the discrete and the dense? Greek and Flemish students' understanding of density. *Learning & Instruction, 21*, 676–685. doi:10.1016/j.learninstruc.2011.03.005.

Vamvakoussi, X., Kargiotakis, G., Kollias, Mamalougos, N. G., & Vosniadou, S. (2003). Collaborative modelling of rational numbers. In B. Wasson, S. Ludvigsen & U. Hoppe (Eds.), *Designing for change in networked learning environments—Proceedings of the International Conference on Computer Support for Collaborative Learning* (pp. 103–107). Dordrecht: Kluwer.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior, 31*, 344–355. doi:10.1016/j.jmathb.2012.02.001.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*, 453–467. doi:10.1016/j.learninstruc.2004.06.013.

Vamvakoussi, X., & Vosniadou, S. (2007). How many numbers in an interval? Presuppositions, synthetic models and the effect of the number line. In S. Vosniadou. In A. Baltas & X. Vamvakoussi (Eds.), *Reframing the conceptual change approach in learning and instruction* (pp. 267–283). Oxford: Elsevier.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*(2), 181–209. doi:10.1080/07370001003676603.

Vamvakoussi, X., & Vosniadou, S. (2012). Bridging the gap between the dense and the discrete: the number line and the "rubber line" bridging analogy. *Mathematical Thinking and Learning, 14*, 265–284. doi:10.1080/10986065.2012.717378.

Vamvakoussi, X., Vosniadou, S., & Van Dooren, W. (2013). The framework theory approach applied to mathematics learning. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (2nd Ed.) (pp. 305–321). New York, NY: Routledge.

Van Dooren, W., Vamvakoussi, X., & Verschaffel, L. (2013). Mind the gap–Task design principles to achieve conceptual change in rational number understanding. In C. Margolinas (Ed.), *Task design in mathematics education: Proceedings of ICMI Study 22* (pp. 519–527). Oxford: International Commission on Mathematical Instruction.

Verschaffel, L., & Vosniadou, S. (Guest Eds.). (2004). The conceptual change approach to mathematics learning and teaching [Special Issue]. *Learning and Instruction*, *14*(5).

Vosniadou, S. (1989). Analogical reasoning as a mechanism in knowledge acquisition: A developmental perspective. In S. Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (pp. 413–436). Cambridge, MA: Cambridge University Press.

Vosniadou, S., Ioannides, C., Dimitrakopoulou, A., & Papademetriou, E. (2001). Designing learning environments to promote conceptual change in science. *Learning and Instruction, 11*, 381–419. doi:10.1016/S0959-4752(00)00038-4.

Vosniadou, S., Vamvakoussi, X., & Skopeliti, I. (2008). The framework theory approach to conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (1st ed., pp. 3–34). Mahwah, NJ: Lawrence Erlbaum Associates.

Yilmaz, S., Eryilmaz, A., & Geban, O. (2006). Assessing the impact of bridging analogies in mechanics. *School Science and Mathematics, 106*(6), 220–230. doi:10.1111/j.1949-8594.2006.tb17911.x.