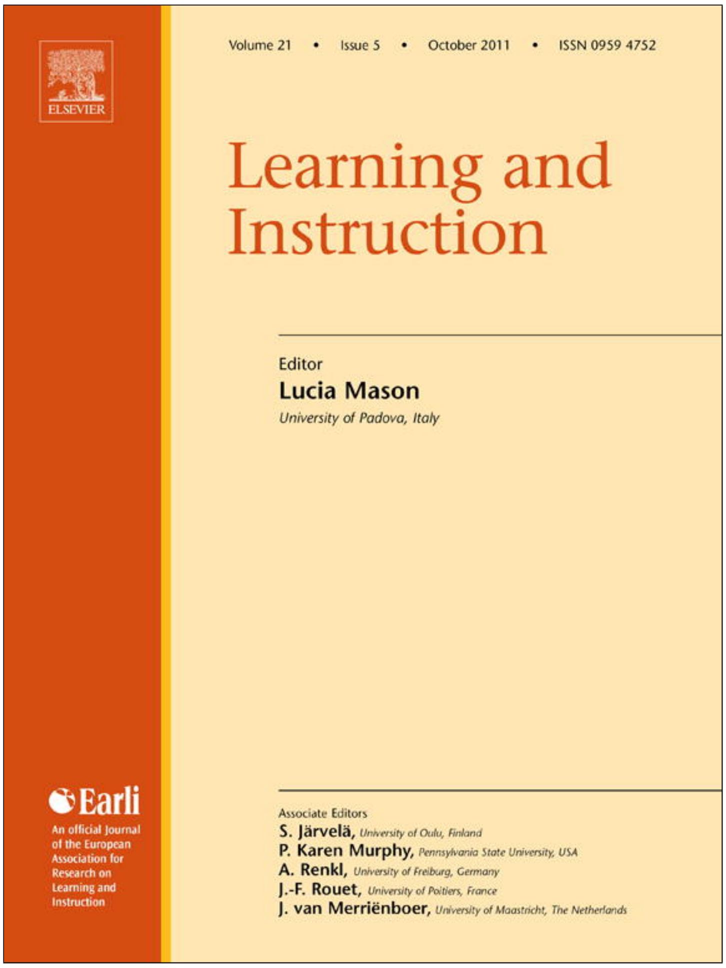
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What fills the gap between discrete and dense? Greek and Flemish students' understanding of density

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Abstract

It is widely documented that the density property of rational numbers is challenging for students. The framework theory approach to conceptual change places this observation in the more general frame of problems faced by learners in the transition from natural to rational numbers. As students enrich, but do not restructure, their natural number based prior knowledge, certain intermediate states of understanding emerge. This paper presents a study of Greek and Flemish 9th grade students who solved a test about the infinity of numbers in an interval. The Flemish students outperformed the Greek ones. More importantly, the intermediate levels of understanding—where the type of the interval endpoints (i.e., natural numbers, decimals, or fractions) affects students' judgments—were very similar in both groups. These results point to specific conceptual difficulties involved in the shift from natural to rational numbers and raise some questions regarding instruction in both countries.

Keywords: Conceptual change; Rational numbers; Density property

1. Introduction

The present study replicates a study by Vamvakoussi and Vosniadou (2010), conducted with Greek students, albeit with Flemish students. Vamvakoussi and Vosniadou investigated secondary school students' understanding of the dense ordering of rational numbers from the perspective of the framework theory approach to conceptual change (Vosniadou, Vamvakoussi, & Skopeliti, 2008). The density property of the rational numbers differentiates them from natural numbers: The latter are discrete, in the sense that every natural number has a unique successor. Thus, between any two natural numbers there is a finite number—possibly zero—of intermediate numbers. In contrast, between any two non-equal rational numbers there are infinitely many numbers, and one can no longer speak of a successor for a given rational number.

It is widely documented that density is a difficult notion for students to comprehend. A repeated finding is that students of various ages state that there is a finite number of numbers between two rational numbers or point to the successor of a particular rational number (Giannakoulias, Souyoul, & Zachariades, 2007; Hartnett & Gelman, 1998; Hannula, Pehkonen, Maijala, & Soro, 2006; Malara, 2002; Merenluoto & Lehtinen, 2002; Tirosh, Fischbein, Graeber, & Wilson, 1999; Vamvakoussi & Vosniadou, 2004, 2007). Such findings indicate that students mistakenly assign the property of discreteness to rational numbers. Thus students' difficulties with density can be placed in the more general framework of the problems faced by learners in the transition from natural to rational and real numbers. This transition is characterized by the interference of natural number knowledge in rational number tasks, where relying on natural number reasoning is no longer appropriate and leads to systematic errors. This phenomenon is acknowledged by mathematics educators and cognitive-developmental psychologists as an instance of a situation where prior knowledge hinders—rather than

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facilitates—further learning. This calls for substantial changes to students' conceptual organization of number, namely conceptual change (Desmet, Grégoire, & Mussolin, 2010; Gelman, 2000; Hartnett & Gelman, 1998; Merenluoto & Lehtinen, 2002; Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004, 2007).

1.1. Theoretical background

In explaining how prior knowledge of number hinders further learning, Vamvakoussi and Vosniadou (2010) advocate the framework theory approach to conceptual change. Based on evidence from cognitive-developmental research, this theoretical frame assumes that young children organize their everyday experiences in the context of lay culture in domain-specific conceptual structures, termed framework theories. These initial theories constitute explanatory frameworks that are generative: They underlie children's predictions and explanations regarding unfamiliar situations in a relatively coherent way. Thus, with respect to different perspectives on conceptual change, the framework theory approach emphasizes *coherence* rather than *fragmentation* of children's thinking (for a thorough discussion, see Disessa, 2006; Vosniadou et al., 2008). The framework theory approach to conceptual change was originally developed to account for the challenges that students face in regard to the learning of certain concepts in science (Vosniadou et al., 2008). It is assumed that a major source of difficulty is the incompatibility of the background assumptions of students' initial theories and the scientific ideas, to which they are exposed mainly via instruction. A prominent example is students' initial understanding of the Earth as a flat, motionless, physical object where gravity works in an up-down direction. This understanding is in sharp contrast with the Earth as an astronomical body, which is spherical, rotates upon its axis, and revolves around the Sun. Vosniadou and her colleagues have identified a small number of conceptions of the shape of the Earth that are *synthetic* in the sense that they incorporate some aspects of the new, scientifically correct information while the pre-existing, underlying assumptions are not re-evaluated. One such example is the conception of Earth as a flat disc, which combines the flatness with elements of "roundness".

In the past few years, the conceptual change perspective on learning has been fruitfully applied in the domain of mathematics (e.g., Greer & Verschaffel, 2007; Prediger, 2008; Verschaffel & Vosniadou, 2004). Regarding the development of the number concept, the framework theory approach to conceptual change assumes that, before they are exposed to rational number instruction, students have formed a rather coherent domain-specific, naïve theory of number. This theory shapes their expectations about what counts as a number and how numbers are supposed to behave. From the students' point of view, numbers are essentially discrete *counting* numbers that obey the successor principle and are grounded in additive reasoning (Vosniadou et al., 2008; see also Gelman, 2000; Ni & Zhou, 2005; Smith et al., 2005). Students are of course not expected to have an elaborated and explicit understanding of natural numbers and their properties. The term *theory* is employed to denote a complex system of interrelated ideas and beliefs, which is relatively coherent and acts as a framework underlying students' thinking in a given (problem) situation.

An essential hypothesis of the framework theory approach to conceptual change in the domain of number is that there are certain intermediate states of understanding that create a bridge between the initial perspective of number and the intended one that is as yet unavailable to the student. These intermediate states are generated as students enrich, via the use of additive learning mechanisms, their knowledge base with new information about numbers provided through instruction without actually (or entirely) re-evaluating the underlying assumptions of their initial theory of number. At the secondary level, taking a rational number perspective requires a) conceptualizing natural and non-natural numbers as members of the same family, b) differentiating between numbers and their symbolic representation, and c) being aware that rational numbers behave differently than natural numbers (e.g., with respect to ordering and operations). Given these aspects, Vamvakoussi and Vosniadou (2010) described a plausible path regarding the development of rational number understanding in school settings: Starting from a conceptualization of numbers consistent with natural numbers, students are introduced to non-natural numbers in the form of decimals and fractions. As they rely heavily on their initial understandings of number to interpret information about these new constructs, it was hypothesized that natural number properties are attributed to rational numbers (see also Moss, 2005; Ni & Zhou, 2005; Smith et al., 2005). At the same time, students face difficulties in conceptualizing natural and non-natural numbers as a unified system of numbers, invariant under different symbolic representations (see also Kilpatrick, Swafford, & Findell, 2001; Markovits & Sowder, 1991). Nevertheless, they do enrich their knowledge base with new information on decimals and fractions. It was thus hypothesized that a specific *synthetic conception* would be the conceptualization of rational numbers as a loose collection of different, unrelated "sets" of numbers (i.e., natural numbers, decimals, and fractions). These hypotheses were tested in the context of students' understanding of density. Density was identified as a notion with the potential to reveal students' difficulties in overcoming an essential presupposition of their assumed initial theories of number, namely discreteness. More importantly, it was predicted that there would be intermediate states of understanding of this notion, allowing for natural numbers, decimals, and fractions to behave differently with respect to ordering (discrete/dense).

Focusing on secondary students' judgments regarding the number of intermediate numbers in an interval, Vamvakoussi and Vosniadou (2004, 2007) obtained qualitative and quantitative evidence that confirmed the above predictions. In their most recent study (Vamvakoussi & Vosniadou, 2010) they systematically investigated this phenomenon with Greek 7th, 9th, and 11th grade students. The idea of discreteness was present in all age groups. In addition, the type of interval endpoints (i.e., natural numbers, decimals, or fractions) had a large effect on students' judgments regarding the number as well as the type of numbers in a given interval. Specifically, students were more inclined to answer that there is an infinite number of intermediates in the case of natural numbers, but less inclined in the case of decimals and fractions. Moreover, students were reluctant to accept that there can be decimals between fractions and vice versa.

There are some findings in the literature which indicate that the interval endpoints have an effect upon students' answers regarding the number as well as the type of intermediate numbers (see for example Tirosh et al., 1999). However, this study was—to the best of our knowledge—the first to investigate these two phenomena systematically and in combination. Vamvakoussi and Vosniadou suggested that their study provided supportive evidence for the framework theory approach to conceptual change: They identified intermediate states of understanding of density, where students were apt to consider the infinity of numbers in an interval in some but not all cases. This indicated that the idea of discreteness is robust and that the understanding of density is not an "all or nothing" situation. As expected, students in these intermediate states of understanding did not respond randomly. Rather, their responses depended on the interval endpoints. This finding was in line with the prediction that different types of numbers (i.e., natural numbers, decimals, or fractions) would not be treated in the same way with respect to ordering (dense/discrete). This was interpreted as a manifestation of the synthetic conception of the rational numbers set as a collection of unrelated sets instead of a unified number system.

The original study was conducted on a sample of Greek students, who came from middle class state-funded schools in an urban area. A replication of that study with a different population would test the generality of its results. In addition, Vamvakoussi and Vosniadou relate their findings to a theoretical stance that makes a more general claim about the processes of learning counter-intuitive concepts in science and mathematics. Thus, a replication of their study is of theoretical interest as well: If a different population did not appear to face any difficulty with the particular notion, or did not show intermediate states comparable to those observed amongst the Greek students, one could argue that the results of the original study were merely a by-product of instruction in the context of a particular educational system. Similarly, if students from a different population either succeeded or failed across all number types or made random mistakes, then not only the generality of the particular findings but also the hypotheses stemming from the framework theory approach to conceptual change could be questioned.

1.2. Educational background

For the replication study, we compared the results of Flemish (i.e., Dutch-speaking Belgian) students and Greek students in the 9th grade. We identified 9th grade students as an age group of particular interest: According to the curricula of both countries, 9th grade students have been taught everything that is in principle required to complete the research tasks correctly. In addition, 9th grade students were an interesting age group in the original study: They performed significantly better than the younger participants (7th grade students) and just as well as the older ones (11th grade students). Finally, 9th grade students are the age group targeted by PISA (Programme for International Student Assessment). Therefore, there is further information available regarding the two populations' mathematical competencies.

In Greece as well as in Flanders, compulsory education extends at least up to the 9th grade. With the exception of a small number of students who attend specialized high schools emphasizing musical or athletic training, the vast majority of the Greek students attend the same type of lower high school (General Gymnasium) and follow the same curriculum in mathematics. All 9th grade students attend 4 h of mathematics instruction per week and use the same textbook.

In contrast, Flemish students in general secondary education choose between a variety of study streams from the 9th grade onwards. In all study streams, students attend 4 h of mathematics instruction per week, with the exception of the students who choose the science-oriented stream. These attend 6 h of mathematics instruction per week. The same national standards apply to all students.

The mathematics curricula of Greece and Flanders do not differ substantially in terms of content, at least with respect to number-related material. By grade 9, both Flemish and Greek students have been introduced to the terms *natural*, *rational*, *irrational*, and *real* numbers. In addition, they have already had extensive practice in procedural aspects of the number concept, including ordering, operations, conversions from decimal to fractional form, and vice versa. Flemish students are introduced to various types of non-natural numbers slightly earlier than their Greek peers. They encounter fractions in the 2nd grade and negative numbers already at the elementary level. Greek students, on the other hand, encounter fractions and negative numbers in the 3rd grade and at the lower secondary level, respectively. Flemish students are introduced to the notion of rational number in the 7th grade, one year earlier than their Greek peers. It should be noted that the 7th grade textbook of the Flemish participants of our study made use of Venn diagrams to make the interrelations between the various subsets of rational numbers explicit. In contrast, Venn diagrams are presented in Greek textbooks in the 10th grade. In a more general fashion, despite its overarching orientation toward the use of numbers in real life situations and problem solving, the Flemish curriculum pays explicit attention to the conceptualization of decimals and fractions as *numbers*. For example, the number line is used as a means to present a unified view of natural numbers and decimals. Specifically, it is explicitly mentioned that decimals are numbers and, as such, they can be placed on the number line. Moreover, the conversion of fractions to decimals is explicitly used to justify the claim that fractions are numbers. In contrast, this issue is not explicitly dealt within the Greek textbooks, which stem from a procedure-oriented curriculum.

Density is not explicitly mentioned in the curricular goals of either of the two countries. Nevertheless, tasks that implicitly relate to this notion, such as the approximation of irrational numbers by rational ones, are present in both curricula.

Belgium and Greece participated in all three cycles of PISA, which took place in 2000, 2003, and 2006. Greek students have been consistently ranked below the OECD average for mathematics, whereas Belgian students have been ranked above the OECD average. In particular, Flemish students were ranked among the best performing students worldwide (all information can be accessed at http://pisacountry.acer.edu.au/).

1.3. Hypotheses

Given the above considerations, we expected that a population of Flemish students from a similar background to the Greek population in the original study (i.e., attending middle-class, urban state-funded schools) would perform better than the Greek ones in the tasks employed by Vamvakoussi and Vosniadou in their study (2010) (Hypothesis 1). On the basis of the framework theory approach to conceptual change, we expected that Flemish students would nevertheless face similar conceptual difficulties with the research tasks as their Greek peers. We hypothesized that they would be achieving intermediate levels of understanding where the idea of discreteness is overcome for some but not all types of numbers. More specifically, we predicted that the interval endpoints would have a similar effect on Flemish and Greek students' judgments in terms of mean performances (Hypothesis 2), and also in terms of individual students' responses across the different number types (Hypothesis 3). Finally, we hypothesized that the idea of discreteness would be robust in the Flemish population as well, in the sense that there would be students who would consistently state that there is a finite number of intermediates across all types of numbers (Hypothesis 4).

2. Method

2.1. Participants

The participants of this study were 84 Greek and 128 Flemish 9th grade students. They came from middle-class, state-funded schools (two Greek and three Flemish), situated in the areas of Athens and Leuven, respectively. For each school, all 9th grade classes participated in the study. All Greek students came from General Gymnasia, and the various study streams available to the Flemish students were representatively present in the Flemish portion of the sample.

2.2. Materials

We used the questionnaire developed by Vamvakoussi and Vosniadou (2010), which was translated first from Greek to English, and then to Dutch. It was a multiple-choice questionnaire that consisted of 14 items asking how many numbers there are between two given rational numbers. The items—which were offered in a randomized order—belonged to four item blocks based on the type of the interval endpoints, namely two natural numbers (Nn, 2 items), a natural number and a decimal (NnDec, 4 items), two decimals (Dec, 4 items), and two fractions (Fra, 4 items). The specific pairs of numbers included in the questionnaire are presented in Table 1.

The answering alternatives offered were the same across all items and were as follows

a) There is no other number
b) There is a finite number of decimals
c) There is a finite number of fractions
d) There are infinitely many decimals
e) There are infinitely many fractions
f) There are infinitely many numbers and they can take different forms, such as decimals, fractions, and non-terminating decimals
g) I don't agree with any of the above. I believe that …

The expression "a finite number of" was explained as "a specific number of numbers that could be written down one by one". Alternative (a), coded as Fin_0, corresponds to the most "naïve" answer, reflecting the idea that the given numbers are successive. Choosing the alternatives (b) or (c), coded as $Fin_{\neq 0}$/*specific type*, means that the given numbers are not deemed successive but that the intermediate numbers are still finite in number and also of a specific type (i.e., either decimals or fractions). Alternatives (d) and (e), coded as Inf-/*specific type*, are more advanced than the preceding ones in that they allow for infinitely many intermediate numbers. The intermediates, however, are again of the same type as the interval endpoints. Alternative (f) is the most sophisticated answer, coded as Inf.

Finally, alternative (g) offered students the option of providing an answer different to the ones already presented. We note in advance that very few students chose this alternative. These students either gave more examples of types of intermediate numbers (e.g., irrational numbers) or accepted the alternatives (d), (e), and (f), explaining that (e) and (d) follow from (f). They were all credited with a correct answer (Inf). The Fin_0 answer was scored as 1, the $Fin_{\neq 0}$ as 2, the Inf- as 3, and the Inf as 4. Cronbach's α for the 14 items was .953.

2.3. Procedure

The questionnaires were administered during regular school hours in the classrooms. One of the researchers read aloud

Table 1
The number pairs included in the questionnaire.

Item block			
Nn	NnDec	Dec	Fra
0 and 1	.9 and 1	2.4 and 2.5	3/5 and 4/5
99 and 100	6 and 6.1	.1 and .2	1/3 and 2/3
	7 and 7.001	.005 and .006	1/7 and 1/6
	.009 and 1	3.124 and 3.125	1/8 and 1/7

some clarifications/instructions, which were also written on the first page of the questionnaire. These were the following: a) "*The term 'numbers' refers to real numbers -all numbers that you know of are real numbers*", b) "*We say that a number is* ***between*** *two other numbers, if it is greater than the first and smaller than the second",* c) "*The particular multiple choice responses used in the questionnaires were provided by students of your age. Do not hesitate to express a different opinion, because it might be better or more accurate than the given answers*", and *d)* "*You can choose only one of the answers*". In case they found that more than one of the given answers fitted, students were instructed to use the open-ended alternative (g) to express and explain their opinion. The students had 45 min to complete the questionnaire.

3. Results

3.1. Comparisons by nationality

Students' total mean performance in the 14 items (Greek: $M = 2.78$, $SD = .82$, Flemish: $M = 3.04$, $SD = .83$, on a maximum of 4) was subjected to an analysis of variance with nationality as between-subjects factor. The results showed a weak, albeit significant, main effect for nationality, $F(1, 211) = 4.981$, $p < .05$, $\eta^2 = .023$ (Hypothesis 1).

3.2. Comparisons across the different number types by nationality

Students' mean performances in Nn, NnDec, Dec, and Fra were subjected to a repeated measures analysis of variance with number type as within-subject factor and nationality as between-subjects factor. The results showed main effects for number type, $F(3, 630) = 23.490$, $p < .001$, $\eta^2 = .101$, and nationality, $F(1, 210) = 2676.336$, $p < .05$, $\eta^2 = .022$; and no interaction effect between number type and nationality. Table 2 shows that Greek and Flemish students' mean performances across the different number types follow a very similar pattern (Hypothesis 2).

Specifically, for both groups, mean performance was higher for Nn items compared to all other types of items, particularly the Dec and Fra items. Mean performance gradually decreased for NnDec and Dec items and was lowest for Fra items. Table 2 also shows that Flemish students performed better than their Greek peers across all types of items, the difference being more salient for Dec and Fra items.

Table 2
Mean, standard error, and 95% confidence interval of performance by number type and nationality.

Nationality	Number Type	M	SE	95% Confidence Interval	
				Lower Bound	Upper Bound
Greek	Nn	3.11	.10	2.92	3.30
	NnDec	2.77	.10	2.57	2.96
	Dec	2.74	.10	2.53	2.94
	Fra	2.69	.10	2.49	2.89
Flemish	Nn	3.29	.08	3.13	3.44
	NnDec	3.02	.08	2.86	3.18
	Dec	3.02	.08	2.85	3.18
	Fra	2.98	.08	2.82	3.13

3.3. Student profiles

Both Greek and Flemish students performed better for natural than for non-natural numbers. This finding already suggests that the type of interval endpoints affects students' judgments regarding the infinity of numbers in an interval (Hypothesis 2) and thus also the existence of synthetic conceptions. It is, however, based on the mean performance of our sample across the different number types, and as such it does not provide information about the individual student's treatment of the different number types (Hypothesis 3). To examine Hypothesis 3 and Hypothesis 4, we formed individual student profiles.

3.3.1. Cluster analysis

To obtain individual student profiles, Vamvakoussi and Vosniadou examined the number of correct responses within each number type (i.e., natural numbers, decimals, and fractions) and used an a priori criterion to place students in different categories. In this study, we opted for a very different approach. Specifically, we employed two-step cluster analysis in order to derive categories directly from the data. In the first stage of this procedure, the records are clustered into many small sub-clusters. Then these sub-clusters are arranged into an appropriate number of clusters. SPSS 19 offers a choice of statistics that can be applied to determine the number of clusters which provides the best description of the data.

Before running the cluster analysis, we transformed the data set in two ways. First, we were primarily interested in students' responses *within* each type of number (i.e., natural numbers, decimals, and fractions). Therefore we excluded, similarly to Vamvakoussi and Vosniadou (2010), the NnDec items from our analysis. Second, we rescored our data so that Fin_0 and $Fin_{\neq 0}$ were collapsed into one response type, namely the FIN one ("there is a finite number of numbers in the interval, either 0 or $\neq$ 0"). This is because both responses reflect the idea of discreteness. Moreover, the great majority of students (65.1%) gave no Fin_0 responses across the items and another 25.5% gave at most three. Only 1.8% gave more than five Fin_0 answers. The FIN response type was scored as 1. The Inf- and Inf response types were scored as 2 and 3, respectively, leading to a maximum mean performance of 3.

We conducted a two-step cluster analysis on nine variables representing the number of FIN, Inf-, and Inf answers that the individual student gave for each type of interval endpoints. For example, the values 2, 1, 1 for fractions corresponded to a response of 2 FIN answers, 1 Inf- answer, and 1 Inf answer in this particular block. We employed the log-likelihood algorithm, and we determined the optimal solution (in terms of the number of clusters) using the Schwartz Bayesian Information Criterion (BIC), a goodness-of-fit measure wherein smaller values indicate a better fit. Table 3 provides information about the BIC indices for the adjacent solutions.

Table 3
BIC indices for 15 adjacent solutions of the cluster analysis.

Number of Clusters	Schwarz's Bayesian Information Criterion (BIC)	BIC Change	Ratio of BIC Changes	Ratio of Distance Measures
1	4453.352			
2	3605.340	−848.011	1.000	2.013
3	3265.063	−340.277	.401	1.997
4	3174.837	−90.226	.106	1.154
5	3118.090	−56.747	.067	1.401
6	3123.558	5.467	−.006	1.377
7	3171.532	47.975	−.057	1.036
8	3223.449	51.917	−.061	1.006
9	3276.056	52.607	−.062	1.105
10	3338.973	62.918	−.074	1.086
11	3409.636	70.662	−.083	1.157
12	3492.484	82.848	−.098	1.048
13	3578.895	86.411	−.102	1.220
14	3678.695	99.800	−.118	1.025
15	3779.974	101.278	−.119	1.140

Note. The changes are from the previous number of clusters in the table. The ratios of changes are relative to the change for the two cluster solution. The ratios of distance measures are based on the current number of clusters against the previous number of clusters.

The BIC value reached its minimum in a solution with five clusters, which was selected as the optimal one. This solution explained a substantial percentage of the variance in our data, namely 68.7%. Fifty-four students (25.5%) were placed in the first cluster, 32 (25.1%) in the second, 34 (16.0%) in the third, 32 (15.1%) in the fourth, and 60 (28.3%) in the fifth.

Table 4 presents the frequencies and percentages of the response types in the total of non-null answers provided within each cluster. The great majority of responses in Cluster 1—hereafter called *Finiteness*—was FIN. This response was substantially lower but remained dominant in Cluster 2, hereafter *Advanced Finiteness*. In the two subsequent clusters the great majority of responses were of the "infinitely many intermediates" type. However, in Cluster 3—hereafter *Naïve Infinity*1—the dominant answer was Inf-, that is, the infinite amount of intermediate numbers was mostly deemed to be of the same type as the interval endpoints. On the contrary, the dominant answer in Cluster 4—hereafter *Naïve Infinity 2*—was Inf. In spite of this difference, a common feature of these two clusters was the presence of a considerable number of FIN answers. In Cluster 5—hereafter *Sophisticated Infinity*—practically all answers were Inf.

Table 4
Frequency and percentage of response types by cluster.

Cluster	Response type			
	FIN	Inf-	Inf	Total
1− *Finiteness* ($N = 540$)	476	25	36	537
	88.1%	4.6%	6.7%	99.4%
2 − *Advanced Finiteness* ($N = 320$)	123	98	94	315
	38.4%	30.6%	29.4%	98.4%
3− *Naïve Infinity 1* ($N = 340$)	84	213	43	340
	24.7%	62.6%	12.6%	100.0%
4 − *Naïve Infinity 2* ($N = 320$)	104	65	145	314
	32.5%	20.3%	45.3%	98.1%
5 − *Sophisticated Infinity* ($N = 600$)	1	0	599	600
	.2%	.0%	99.8%	100.0%

3.3.2. Cluster features

Table 5 presents the means, standard errors, and 95% confidence intervals with respect to students' performance for Nn, Dec, and Fra by cluster. It also presents the percentage of the response types within each number type. Students in *Finiteness* clearly performed worse than students in all other clusters in terms of total mean performance as well as within all number types. Within this cluster, natural numbers elicited higher mean performances than decimals and fractions. This is due to the fact that students in *Finiteness* gave FIN responses across all number types, but relatively more Inf answers for natural numbers.

Students in *Sophisticated Infinity* gave Inf responses across all types of numbers. As a result, mean performance over the different number types was clearly higher in this cluster than in all other clusters, with the exception of *Naïve Infinity 2* in the case of natural numbers. Performance for natural numbers differentiated the three intermediate clusters, increasing gradually from one cluster to the next. A common feature of these three clusters is that mean performance for fractions was the lowest of all number types.

Advanced Finiteness and *Naïve Infinity 1* cannot be differentiated in terms of total mean performance or mean performance for decimals and fractions. However, considering the percentages of responses types for Nn, Dec, and Fra presented in Table 5, it becomes clear that the pattern of responses was quite different for these two clusters. Specifically, the FIN response was dominant in *Advanced Finiteness* across all number types. Students in this cluster also provided a mixture of Inf- and Inf answers. On the other hand, the Inf-response was dominant across all number types for students in *Naïve Infinity 1*. Within this cluster, there is a salient difference between natural numbers and decimals and between natural numbers and fractions. One can also notice that fractions elicited a considerable number of FIN answers.

Naïve Infinity 2 is characterized by the absence of FIN and Inf- answers for natural numbers. The response Inf is also dominant for decimals. As a result, mean performance for decimals differentiates this cluster from all preceding ones. In the case of fractions, however, the dominant response is FIN. In fact, this cluster is quite similar to *Naïve Infinity 1* with respect to mean performance for fractions.

3.3.3. Distribution of students across the clusters by nationality

Table 6 shows the distribution of Greek and Flemish students across the clusters. Greek students were distributed more amongst the first two clusters. On the other hand, Flemish students were more heavily represented in the final cluster. A chi-square test, however, showed that this difference was not significant, $\chi^2(4, N = 212) = 6.920$, $p > .05$. Thus, despite their overall better performance, Flemish students were also represented in the two first clusters, wherein the FIN answer was dominant across all number types (Hypothesis 4).

Table 5
Mean, standard error, and 95% confidence interval of performance; and percentage of response types by number type and cluster.

Cluster	Number Type	M	SE	Confidence Interval		Percentages of Response Types		
				Lower Bound	Upper Bound	FIN	Inf-	Inf
1 *Finiteness*	Nn	1.36	.06	1.24	1.48	81.5	.9	17.6
	Dec	1.10	.05	1.00	1.19	83.3	2.3	4.2
	Fra	1.16	.06	1.04	1.27	87.0	8.8	3.7
	Total	1.18	.04	1.10	1.25			
2 *Advanced Finiteness*	Nn	1.90	.08	1.73	2.05	40.6	29.7	29.7
	Dec	1.90	.06	1.77	2.02	37.5	30.5	30.5
	Fra	1.85	.08	1.70	2.00	38.3	31.3	28.1
	Total	1.88	.05	1.78	1.98			
3 *Naive Infinity 1*	Nn	2.22	.08	2.07	2.37	11.8	54.4	33.8
	Dec	1.88	.06	1.76	2.01	16.9	77.9	5.1
	Fra	1.71	.07	1.56	1.85	39.0	51.5	9.6
	Total	1.88	.05	1.79	1.97			
4 *Naive infinity 2*	Nn	2.91	.08	2.75	3.06	.0	.0	96.9
	Dec	2.15	.06	2.02	2.27	35.2	14.8	50.0
	Fra	1.63	.08	1.48	1.77	46.1	35.9	14.8
	Total	2.10	.05	1.99	2.19			
5 *Sophisticated Infinity*	Nn	2.99	.06	2.87	3.10	.8	.0	99.2
	Dec	3.00	.05	2.91	3.10	.0	.0	100.0
	Fra	3.00	.06	2.89	3.11	.0	.0	100.0
	Total	3.00	.04	2.93	3.07			

Interestingly, only half of the 34 Flemish students with the stronger mathematical background were placed in *Sophisticated Infinity.* From the remaining half, 4 (11.8%) were placed in *Naïve Infinity 2*, 6 (17.6%) in *Naïve Infinity 1*, 2 (5.9%) in *Advanced Finiteness*, and 5 (14.7%) in *Finiteness.*

4. Discussion

4.1. Summary and discussion of the results

As expected, Flemish students outperformed their Greek peers in the tasks focused on the infinity of numbers in an interval (Hypothesis 1). However, these tasks were far from trivial for Flemish students, including those that followed the science study stream and were therefore considered to have a stronger mathematical background.

Approximately a quarter of our participants were included in *Finiteness*. These students did not differentiate between natural and rational numbers with respect to ordering and transferred the property of discreteness from natural to non-natural numbers. Interestingly, more than half of both the Flemish and Greek students were not consistently on the "finite" or on the "sophisticated infinite" side. Students in *Advanced Finiteness* provided a substantial number of Inf and Inf- answers. Nevertheless, FIN answers were still dominant across all number types. Although they did consider the infinity of intermediates in an interval, these students were still constrained by the idea of discreteness and were reluctant to accept that the intermediates can be of various symbolic representations.

The effect of the interval endpoints was clear in *Naïve Infinity 1* and *Naïve Infinity 2*. Students in *Naïve Infinity 1* provided a large number of Inf- answers, indicating that they were affected by the interval endpoints as far as the type of intermediate numbers was concerned. Moreover, there was an endpoint effect with respect to the number of intermediate numbers. These students provided more Inf answers for natural numbers than for decimals and fractions. On the other hand, fractions elicited considerably more FIN answers than decimals. A similar pattern can be observed in *Naïve Infinity 2*, which is characterized by a practically flawless performance for natural numbers. Inf answers were also dominant for decimals. On the contrary, FIN answers were dominant for fractions. Moreover, when students considered the infinity of intermediates in this case, they opted for numbers in the form of fractions.

These findings corroborate the claim that understanding of density is not an "all or nothing" situation. It is evident that the idea of discreteness was robust (Hypothesis 4) and persisted even when students considered the infinity of intermediates in an interval. As expected, the type of interval endpoints used affected students' judgments regarding the type as well as the

Table 6
Frequency and percentage of students by cluster and nationality.

Cluster	Nationality		Total
	Greek	Flemish	
1– *Finiteness*	25	29	54
	29.8%	22.7%	25.5%
2 – *Advanced Finiteness*	17	15	32
	20.2%	11.7%	15.1%
3– *Naïve Infinity*1	12	22	34
	14.3%	17.2%	16.0%
4 – *Naïve Infinity 2*	13	19	32
	15.5%	14.8%	15.1%
5 – *Sophisticated Infinity*	17	43	60
	20.2%	33.6%	28.3%
Total	84	128	212
	100.0%	100.0%	100.0%

number of intermediate numbers. This effect was evident both at the group and at the individual level (Hypotheses 2 and 3). Similar to the findings of Vamvakoussi and Vosniadou (2010), it appears that the idea of an infinity of numbers in an interval was more accessible to students in the case of natural numbers, and less so in the case of fractions. This holds for Greek and Flemish students alike, considering that the distribution of the two nationalities in the intermediate clusters, in particular in Naïve Infinity 1 and Naïve Infinity 2, was very similar.

These findings replicate those of the original study (Vamvakoussi & Vosniadou, 2010) in a different population, providing supporting evidence for their generality. It should also be stressed that the individual profiles in our study were obtained via a cluster analysis, in which statistical criteria were used to determine the number of clusters that was best fitting to the data. The profiles obtained bear remarkable similarities to the categorization that was applied by Vamvakoussi and Vosniadou, even though the latter categorization was developed on the basis of a priori criteria. The observed similarities constitute more convincing evidence for specific challenges that students meet in acquiring the rational number concept than either of the two studies could offer alone.

Furthermore, the finding that students' responses depended on the interval endpoints could be interpreted as an indication that they treated natural numbers, decimals and fractions as different kinds of numbers, as opposed to instances or alternative representations of rational numbers. In terms of the framework theory approach to conceptual change, this is a *synthetic conception* of the rational number set.

Therefore, the results of this study are in line with the predictions derived from the framework theory approach to conceptual change (Vamvakoussi & Vosniadou, 2010; Vosniadou et al., 2008). Thus, this theoretical frame could be viewed as a source of fruitful hypotheses on students' learning in the domain of number.

4.2. Considerations regarding instruction

This study raises some questions in regard to rational number instruction, in view of the fact that Flemish students, placed by PISA among the best students worldwide with respect to their mathematics competence, appeared to face similar conceptual difficulties to the Greek students with respect to the rational number concept. One could argue that dealing with the notion of density was bound to be difficult for students in both countries, as it is not explicitly taught in either. But it is precisely this fact that makes density tasks appropriate as *generative* tasks: They present students with a novel problem for which they probably do not have a ready-made answer in advance, while they have in principle the required knowledge to attain the correct answer. Thus, failure to deal with the particular tasks is not interesting per se. The focus is rather upon what students' responses reveal about their rational number reasoning. It appears that 9th grade students in both countries, to a considerable extent, are unaware of the differences between natural and rational numbers, confuse numbers with their representations, and do not deem natural and non-natural numbers to be members of the same category, in spite of the fact that they have already been exposed to at least 6 years of instruction upon rational numbers.

The particular difficulties have not gone unnoticed in the rational number development and learning literature (e.g., Kilpatrick et al., 2001; Moss, 2005; Markovits & Sowder, 1991; Ni & Zhou, 2005). We would like, however, to focus on a related but slightly different aspect of the problem. As Greer and Verschaffel (2007) state, within the mathematics education research community there is a conceptual shift away from the view of mathematics learning as enrichment, merely building on prior knowledge without the need for restructuring. This shift, however, has not found its way into the classrooms. Instruction appears to be grounded on the assumption that learning about rational numbers can be accomplished via the gradual accumulation of related information and neglects the fact that prior knowledge may actually stand in the way of further learning (Resnick, 2006; Vosniadou & Vamvakoussi, 2006). Both Greek and Flemish mathematics curricula appear to downgrade the fact that the shift from natural to rational number entails qualitative rather than merely quantitative changes in the concept of number. Thus Greek and Flemish students focus first on natural number arithmetic, validating and strengthening the conception of number as natural number. They are then gradually introduced to non-natural numbers and spend much of their time at elementary school becoming familiar with mostly procedural aspects of these new constructs. In textbooks, the similarities between natural and non-natural numbers are over-emphasized with a view to making the latter more accessible to students. For example, the part-whole aspect of fractions, which allows for natural number reasoning, is prominent. This, however, creates several problems in the long run. The idea that fractions are discrete is arguably one such problem (Mamede, Nunes, & Bryant, 2005; Moss, 2005). On the other hand, a more sophisticated aspect of the similarity between natural and non-natural numbers, namely what makes them members of the same family, is not adequately addressed. In addition, the decimal and fractional representations of rational numbers are mostly presented separately. The 7th grade textbook of our Flemish participants, for example, introduces students to the notion of the decimal number system in the first chapter, attempting to provide a unifying view of natural and rational (non-natural) numbers in decimal form. Fractions, however, are treated separately and as many as four chapters later.

Dealing with the problem of conceptual change in the number concept requires designing instruction on a long-term perspective basis (Greer, 2006). Probably the first step should be to increase the awareness of teachers and textbook developers regarding the problem of conceptual change and the particular difficulties that are bound to appear in the learning process. It seems plausible that teaching that recognizes the specific difficulties that were identified in this paper and explicitly addresses them can lead to considerably better outcomes. On the flip side, being aware of the specific difficulties can help avoiding certain "missteps" in instruction that may actually enhance students' misunderstandings. Let us illustrate such a "misstep": In the Greek textbook, rational numbers are introduced as the set comprising "*all the numbers we have studied so far, namely natural numbers, decimals, fractions, and their respective negative numbers*". Similarly, the Flemish textbook reads "*we call the positive and negative fractions and decimal numbers rational numbers; we will name their set Q*". Such "definitions" attempt to introduce the rational number set in a simple way, building on students' available knowledge. However, they are bound to enhance students' tendency to treat rational numbers as a loose collection of unrelated "sets" of numbers.

It should be noted that the conceptual difficulty relating to the assumed synthetic conception of rational numbers as consisting of different, unrelated "sets" of numbers has consequences for students when dealing with notions that *do* comprise part of their curricula. Consider for instance the notion of the (real) variable, which is in many aspects instrumental in the mathematics curriculum (e.g., with respect to inequalities, the notion of absolute value, functions, etc.). Greek and also Flemish students were found to be reluctant to accept that *any* number—regardless of its type (natural/non-natural) and its symbolic representation (fraction/decimal)—can substitute a variable (Van Dooren, Christou, & Vamvakoussi, 2010). Greek students in particular showed a strong tendency to substitute only natural numbers, which predicted their ability to deal with a series of tasks, such as the solving of equalities (Christou & Vosniadou, 2009).

4.3. Limitations and further research

The purpose of this study has been to test the replicability of the findings of the study undertaken by Vamvakoussi and Vosniadou (2010), also with a view to evaluate the predictions of the framework theory approach to conceptual change in this domain. The results were as expected. However, a study with larger samples and different age groups could form a basis upon which stronger conclusions with regard to the similarities and differences between the populations could be drawn. More importantly, qualitative evidence is necessary to provide more information about the underlying reasoning of students and to verify that students encounter similar obstacles when approaching the tasks. A replication with a population from an educational system with salient differences in respect to the mathematics curriculum would be worth investigating.

A different line of research could focus on intervention studies addressing the specific conceptual difficulties identified in this study. In particular, one could investigate the instructional value of the number line in this respect. The number line is a continuous representation. As such, it has the potential to contradict students' belief that numbers are discrete. In addition, it allows for the integration of different types of numbers within a unified number system (Kilpatrick et al., 2001). Finally, further research could also focus on mathematics teachers, aiming at raising their awareness of students' conceptual difficulties in the shift from natural to rational numbers.

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