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Mind the gap – Task design principles to achieve conceptual change in rational number understanding

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In this paper we focus on the problem of conceptual change in the shift from natural to non-natural numbers. We discuss students' difficulties in this area and present a mathematics textbook analysis to show that this problem is not taken into consideration in instruction. We discuss a number of principles for instruction stemming from the conceptual change perspective to learning and present a number of experimentally tested tasks, designed on the basis of these principles. These tasks were used to investigate and/or to induce conceptual change in the number concept. We argue that such tasks are of value from the point of view of instruction.

Keywords: Rational number understanding, conceptual change

Difficulties in understanding rational numbers

Mastery of the rational numbers represents an important aspect of mathematical literacy. However, learning about rational numbers presents students with many difficulties, mostly when the required reasoning is not in line with their prior knowledge and experience about natural numbers (see Ni & Zhou, 2005, for a review).

In comparing decimals, students judge for instance that longer decimals are larger, thus responding that 2.12 > 2.2 (Resnick et al., 1989); in comparing fractions they think that a fraction gets larger when one of its parts gets larger, resulting in errors such as 2/5 < 2/7 (e.g., Ni & Zhou, 2005). Students also extend the meaning of operations from natural to non-natural numbers. For instance, seeing multiplication of natural numbers as repeated addition leads to the idea that multiplication makes bigger, which has been shown difficult to overcome (Greer, 1994), even in adults (Vamvakoussi, Van Dooren, & Verschaffel, in press). Finally, the dense ordering of rational (and real) numbers is difficult for students to grasp (e.g., Vamvakoussi, Christou, Mertens, & Van Dooren, 2011, Vamvakoussi & Vosniadou, 2004, 2010;)[58].

[58] An order $\leq$ on a set X is dense if, for all x and y in X for which $x < y$, there is a z in X such that $x < z < y$. Unlike the integers, the rational numbers as well as the real number are densely ordered. The real numbers are, in addition, continuous.

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Students initially respond that there are no other numbers between two given pseudosuccessive numbers (e.g., 0.005 and 0.006 or 1/2 and 1/3). Later, they refer to some intermediate numbers, but still do not accept that there are infinitely many.

Theoretical framework

Several researchers have argued that many of students' difficulties with rational numbers can be explained from a conceptual change perspective on learning (e.g., Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005). Within this perspective, we adopt a specific theoretical frame, namely the framework theory approach to conceptual change (FTatCC, Vamvakoussi & Vosniadou, 2010), because it proposes a number of specific and testable principles for the design of instruction and tasks (Greer & Verschaffel, 2007; Tsamir & Tirosh, 2007). Originally developed to account for the challenges that students face in regard to science learning, the FTatCC has been fruitfully applied in the past few years in the domain of mathematics learning (e.g., Greer & Verschaffel, 2007; Verschaffel & Vosniadou, 2004). Based on evidence from cognitive-developmental research, the FTatCC assumes (Vosniadou et al., 2008) that young children organize their everyday experiences in the context of lay culture from an early age in domain-specific conceptual structures, termed *framework theories*. These initial theories constitute explanatory frameworks that are generative: They underlie children's predictions and explanations regarding unfamiliar situations in a relatively coherent way. The incompatibility between the background assumptions of students' initial theories and the scientific ideas to which they are exposed mainly via instruction, is assumed to be a major source of misunderstandings and errors for students. Regarding the development of the number concept, the FTatCC assumes that, before they are exposed to rational number instruction, students have formed a rather coherent domain-specific, naïve theory of number—i.e., a complex system of interrelated ideas and beliefs—based on their extensive experiences in the natural number domain. This theory shapes their expectations about what counts as a number and how numbers are supposed to behave. From the students' point of view, numbers are essentially discrete *counting* numbers and are grounded in additive reasoning (Vamvakoussi & Vosniadou, 2011; see also Ni & Zhou, 2005; Smith et al., 2005).

The FTatCC assumption about the structure and content of students' knowledge of numbers—before they are exposed to non-natural number instruction—implies that the shift from natural to non-natural numbers is a slow and gradual process that is difficult to accomplish and requires substantial instructional support. A factor contributing to this difficulty is that students are typically unaware of the background assumptions of their framework theories and thus do not perceive the necessity to re-evaluate or revise them. The FTatCC predicts that, instead, students enrich via the use of additive learning mechanisms their knowledge base with new incompatible information about numbers provided through instruction, thus creating misconceptions such as the ones described above.

Instruction design principles

The conceptual change perspective on learning and instruction has been traditionally associated with the *cognitive conflict* teaching strategy. This strategy has been subject to criticisms and is now acknowledged as a potentially useful approach provided that it is used with caution and only as one out of several other alternatives (Vosniadou, Ioannides, Dimitrakopoulou, & Papademetriou, 2001). Indeed, there is a number of

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different principles for the design of instruction stemming from the conceptual change perspective on learning (Greer & Verschaffel, 2007; Vosniadou et al., 2001; Vosniadou & Vamvakoussi, 2006; Vosniadou, Vamvakoussi, & Skopeliti, 2008). We will refer here to the ones that are mostly relevant for the purposes of the present paper. We note that, as such, these principles are not unique to the conceptual change perspective on learning, in particular to the FTatCC. We stress, however, that the principles have a very specific focus, namely to address the problems arising from the incompatibility of prior knowledge with the intended new mathematical knowledge. This particular focus has implications also for the implementation of principles in the design of tasks, as we will illustrate in the following sections.

Take students' prior knowledge into consideration

There are several ways for prior knowledge to be taken into consideration in instruction. This principle refers to the necessity to acknowledge the potentially adverse effect of prior knowledge, in cases when it is not compatible with new information coming from instruction. This requires that teachers, curriculum designers, and textbook authors can identify the points where conceptual change is necessary, and that they are informed about students' potential initial understandings. Let us present an example that is related to our discussion in the next section. Prior knowledge and experience about natural numbers can be used to introduce non-natural numbers. In fact, it is commonly used when, for instance, fractions are introduced via their part-whole aspect, or when decimals are presented as whole numbers with a change of units. On the contrary, the differences between natural and non-natural numbers are not explicitly addressed. However, downgrading the differences and focusing on the similarities between natural and non-natural numbers--with a view to build on students' prior knowledge--creates many problems in the long run, as discussed in the first section.

Facilitate students' metaconceptual awareness

As pointed above, students are typically not aware of the background assumptions of their framework theories of numbers (e.g., that numbers are essentially discrete) and this hinders conceptual change learning. It is thus important to create opportunities for students to externalize their ideas, compare them with their peers', and reflect on them. This can be done in learning environments that foster group discussions, particularly when students are engaged in tasks that involve modelling a situation or dealing with external representations. This brings us to the next principle.

Use models and external representations

Again, this principle is not unique to the FTatCC. However, it is important to note that from the FTatCC perspective, taking into consideration students' prior knowledge and how it may influence their interpretations of a situation also applies in the case of models and external representation introduced in instruction. Consider, for example, the (real) number line. It is a powerful representation for numbers, but it is also known to be difficult and even misleading for students. For instance, conceptualizing the number line as a ruler may lead students to believe that there is a finite number of numbers in a given interval.

Foster analogical reasoning

Analogical reasoning, in particular cross-domain mapping, is considered an important mechanism for conceptual restructuring. This is because the comparison between two domains may highlight their common features, reveal unnoticed commonalities, and allow for the projection of inferences from one domain to the other. In the process, re-presentation of one or of both domains may occur to improve the match, which may lead to conceptual restructuring. Consider, for example, the complex interplay between the domain of continuous magnitudes and the domain of number that, in the course of the historical development, resulted in the re-conceptualization of the notion of number, as well as of continuity (Vamvakoussi & Vosniadou, 2012).

In the following, we present a textbook analysis showing that the problem of conceptual change in the number concept is not taken into consideration in instruction. Then we present examples of tasks that are grounded on the above principles and have been experimentally tested with respect to their potential to induce conceptual change learning.

A textbook analysis

A central theme in the task design principles mentioned above is that students need to be pointed explicitly to differences between natural and rational numbers. To see to what extent this currently happens, we did an analysis of the three most frequently used primary school mathematics textbook series in Flanders, Belgium. More specifically, the teachers' manuals from year 2 to 6 were analysed, as these included student materials and several additional clarifications and background information.

The units of analysis were the lines in the teachers' manual that in some way dealt with rational numbers (i.e. fractions, decimals, negative numbers). For every line, it was determined whether and to what extent it made reference to differences between natural and rational numbers, or to similarities between them. In cases when such a difference or similarity was pointed out, it was moreover coded whether this happened in an implicit or in an explicit way. Finally, it was also coded for what aspect (the way to determine the size of a rational number, the effect of operations with rational numbers, the representation of rational numbers, or the density of the rational number system) the difference or similarity was referred to.

The results showed that the textbooks were very comparable in their treatment of rational numbers. With respect to the size of rational numbers, none of the textbooks explicitly referred to the fact that rules that are valid to determine the size of natural numbers do not hold for rational numbers. While most observations referred to differences between both kinds of numbers, they were all implicit. Such an implicit reference is for instance a number line showing the location numbers 0.6, 0.75, and 0.8. Students can derive that even though 75 is larger than 6 and 8, 0.75 is still between 0.6 and 0.8, but it is not explicitly pointed out.

With respect to representations, all textbooks referred to differences merely in implicit ways (such as pointing out that 2/4 = 1/2 without explicitly pointing out that any rational number can have infinitely many different representations).

For the domain of operations, both similarities and differences between natural and rational numbers are pointed out, about two thirds are similarities. An example is that a decimal number like 0.72 is written as 72 tenths in order to do operations with it (such as halving or doubling). Only one textbook explicitly mentions that teachers should explicitly address the idea in students that division will lead to a smaller result, and this happens only at one moment in the fifth year.

Finally, the aspect of the density of the rational numbers is hardly dealt with at all in the three textbook series. In the few cases where it is addressed, this happens

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in an implicit way, mostly by pointing out that an interval between two given numbers on a number line can be "stretched" after which more numbers can be found, as illustrated in Figure 1. It is however not explicitly pointed out that infinitely many numbers can be found in any interval or that the stretching can be infinitely repeated.

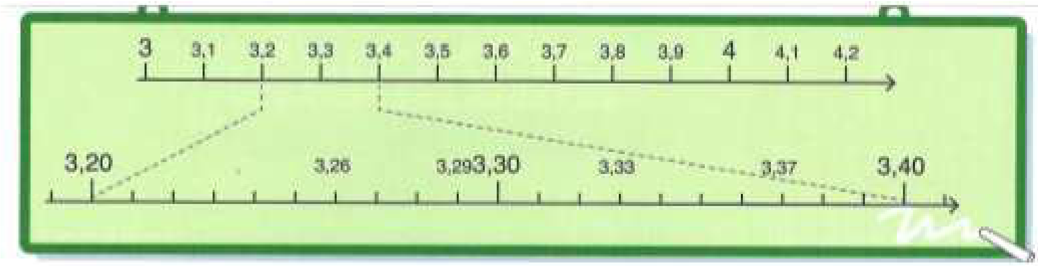

Figure 1. Implicit reference to the density of the number line in a fifth grade textbook

Tasks to investigate and induce conceptual change

In the following we present specific tasks that were employed in experimental settings with a view to investigate secondary students' understandings of the density property of rational (and real) numbers, and to explore possibilities for effective teaching of this counter-intuitive notion. The design of these tasks drew on a series of studies from the FTatCC that focused on students' understanding of density as a paradigmatic case of the problem of conceptual change in the development of the rational number concept (Vamvakoussi & Vosniadou, 2004, 2010; Vamvakoussi et al., 2011). In line with the FTatCC principles, the design of the tasks was informed by empirical evidence about students' pre-existing ideas and typical misconceptions regarding the notion of density in arithmetical, as well as geometrical contexts. This evidence can be summarized as follows: a) the idea of discreteness is robust both in arithmetical and in geometrical contexts, b) students are more inclined to accept that there are infinitely many points on a segment, than that there are infinitely many numbers in an interval, c) accepting that there are infinitely many intermediates (numbers or points) does not imply that one understands that these can never be found one immediately after the other, d) students do not "see" the rational numbers set as a unified system of numbers but rather as consisting of unrelated sets of numbers (e.g., integers, decimals, fractions), which has implications on their judgments about the number, as well as the type of numbers in an interval. In line with the FTatCC principles, the tasks refer to the cross-domain mapping between continuous magnitudes, in particular the straight line, and numbers. This cross-domain mapping is deemed crucial for instruction-induced conceptual change in the number concept (Vamvakoussi & Vosniadou, 2012). It also underlies a representation of numbers, namely the (real) number line that is commonly used in schools settings.

In line with the FTatCC principles, using, evaluating, comparing and constructing representations of numbers and the number line, lie at the heart of the sequence of tasks designed by Vamvakoussi, Kargiotakis, Kollias, Mamalougos, and Vosniadou (2003, 2004) (Table 1). These tasks were experimentally tested in two different settings, both allowing for expressing one's ideas, and discussing and evaluating others' ideas. Specifically, 30 9th graders were split in two groups who worked on the tasks in pairs. Each pair presented their results to their fellow students and they were discussed. One 45-minute session was devoted to each task. Meanwhile, the control group (14 9th graders) worked in their classroom, with paper and pencil, and the results were presented orally and then written on the blackboard by the researcher. The experimental group (16 9th graders) worked in Synergeia, a software designed to support collaborative knowledge building that provides a structured, web-based work space in which documents and ideas can be shared and discussions can be stored. These participants had constant access to their peers' answers, could write comments on them, and respond to comments. Both groups received the same pre- and a post-test with tasks on the density of numbers. They were also interviewed after the intervention. The experimental group improved significantly more in its performance on density tasks than the control group. Moreover, the experimental students displayed greater metaconceptual awareness of the change in their ideas about numbers before and after the intervention. It appears that exchanging ideas on the particular tasks in a structured environment with the features of Synergeia was more profitable for students than the whole class discussion (Vamvakoussi et al., 2003, 2004).

Task	Goal
1. What do you know about the real number line? Describe as good as you can. Read and comment upon the answers of your fellow students.	Express prior knowledge about the real number line
2. We often use the term "the set of real numbers". Suppose someone tries to understand what we mean by that. Could you draw a picture to help him/her understand?	Construct a representation for real numbers
3. We have been talking about two different representations of real numbers: A "formal" one, which we usually use at school, and a second one, which was proposed in our discussion and you seem to find adequate. Could you find a solid reason why we should prefer one over the other?	Compare two different representations
4. Imagine that you can become as small as a point of the number line. Then you could see the other points up close. Suppose that you are on the point that stands for the number 2.3. Can you define what point is the one closest to you? Describe in words or by drawing a picture.	Construct a representation for the number line

Table 1: Working with representations of numbers and the number line: A sequence of tasks

Vamvakoussi and Vosniadou (2012) further elaborated on the cross-domain mapping between numbers and the line. In line with the FTatCC, they designed a text that a) provided explicit information about the infinity of numbers in an interval, b) made explicit reference to the numbers-to-points correspondence, and c) used a bridging analogy (the number line as a rubber line) to convey the idea that points (and numbers) can never be found one immediately after the other. The excerpt regarding the bridging analogy reads:

> The mathematical number line is a strange object. You can imagine it as a rubber band that never breaks, no matter how much you stretch it. Place numbers between 0 and 1, until it looks like you have used all the available points. If you stretch the rubber band, then you will find out that between the points that looked as if there were the one next to the other, there are more available points, corresponding to more numbers. This procedure can be repeated infinitely many times- don't forget that your imaginary rubber band never breaks!

Vamvakoussi and Vosniadou tested experimentally the value of the "rubber-line" text as compared to two other texts that contained the explicit information and

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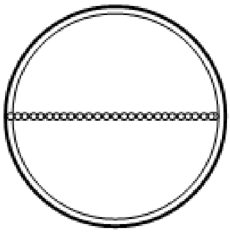
examples of intermediate numbers, or figures illustrating the examples. Six classes of 8th and 11th graders (one experimental class per grade), in total 149 students received a pre-test with density tasks in a arithmetical and a geometrical context, were administered the corresponding text, and then received a post-test containing all the tasks of the pre-test, and 5 additional tasks that examined whether students were able to deal with the no-successor aspect of density (Figure 2). All groups profited from the explicit information about the infinity of numbers presented in the text. However, the experimental group (8th and 11th graders) outperformed the other groups in the "no successor" items of the posttest, and were more consistent in providing correct answers and justifications for their answers.

Kiki says" A line has infinitely many points, which are very close to one another. But if you could magnify a segment of the line at a very high power, then you could see the points one by one, like in the following picture".



➤ **Do you agree with Kiki' s drawing? Yes ❑ No ❑ Explain why:**

Figure 2. Example of a "no-successor" task in a geometrical context

Concluding thoughts

We stress that these tasks come from experimental studies aiming at testing very specific hypotheses, and not in the first place at creating optimal learning environments. As researchers in (the psychology of) mathematics education, being mostly funded to conduct fundamental research, we are not primarily concerned with the development of tasks that can be directly used in classroom teaching, as we mainly aim to analyze and understand students' difficulties in learning particular concepts starting from a certain theoretical stance. Still, we are convinced that our perspective may have an added value for task design. We consider task design an important part of the design of instruction. The tasks presented here were designed on the basis of specific theoretical principles stemming from a conceptual change perspective to learning and instruction. Furthermore, these tasks are empirically tested, also with respect to the conditions under which they can be useful for teachers as well as students.

We suggest that using tasks that prompt students to evaluate, compare, and construct representations (in this case, of numbers) is informative for teachers, in the sense that it provides valuable information on students' thinking. In particular, tasks that were presented as "thought experiments" (e.g., task 4 in Table 1; see also Figure 1) were extremely informative for students' ideas on counter-intuitive notions, which, were not easily accessible via verbal descriptions. On the other hand, the examples presented here indicate that, if such tasks are embedded in a learning environment that is supportive of structured interaction among students, then they may facilitate conceptual change learning. We also provided evidence that even a typical school task, namely extracting information from a text to answer related questions, can lead to conceptual change learning gains, depending on the kind of information that is provided in the text. Specifically, information that bridges between students' initial ideas (e.g., the segment as a "necklace of beads") and the intended mathematical notion (e.g., the segment as a dense array of points) appears to facilitate the grasping of counter-intuitive ideas.

We believe, therefore, that the principles and tasks as elaborated above are interesting from an instructional point of view, and may eventually inspire the development of learning environments. Our textbook analysis – which shows a very large gap with the tasks and principles elaborated above – even strengthens this claim.

Finally, we must stress that the FTatCC is in first instance a cognitively-oriented theory. Of course, students' affect, motivation and beliefs also play an important role in the learning processes to obtain conceptual change, but they are beyond the scope of this paper.

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