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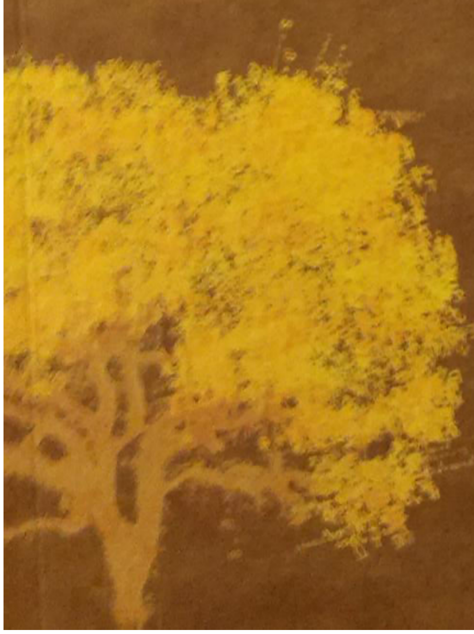

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GREEK AND FLEMISH STUDENTS' UNDERSTANDING OF THE DENSITY OF RATIONAL NUMBERS: MORE SIMILAR, THAN DIFFERENT

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The framework theory approach to conceptual change hypothesizes that in the shift from natural to rational numbers, overcoming the idea that numbers are discrete is gradual and that, in the process, certain intermediate states of understanding will appear in students. This paper presents a study in two different groups of 9th graders, namely Greek and Flemish, who solved a test eliciting their understanding of the infinity of numbers in an interval. The results showed that Flemish students outperformed the Greek ones. However, the intermediate levels of understanding – where the type of the interval end points, i.e., natural numbers, decimals, or fractions, affects students' judgments – were very similar in Greek and Flemish students. These results support the framework theory approach to conceptual change.

THEORETICAL AND EMPIRICAL BACKGROUND

The density property of rational and real numbers is difficult for students to grasp (Giannakoulias, Souyol, Zachariades, 2007; Hannula, Pehkonen, Maijala, & Soro, 2006; Merenluoto & Lehtinen, 2002; Smith, Solomon, & Carey, 2005; Tirosh, Fischbein, Graeber, & Wilson, 1999). A recurrent finding is that students typically answer that there is a finite number of numbers in an interval, or point to the successor of a given rational number. It thus appears that students mistakenly assign the property of discreteness to rational numbers. Vamvakoussi and Vosniadou (in press) investigated secondary students' understanding of density from the perspective of the framework theory approach to conceptual change (Vosniadou, Vamvakoussi, & Skopeliti, 2008). According to this theoretical framework, before students are exposed to rational number instruction, they have already consolidated a *naïve theory* of number which is tied around their knowledge and experience of natural numbers and comprises tacit, background assumptions about what a number is and how it is supposed to behave. The idea that numbers are discrete is a fundamental presupposition of students' initial *theories* of number (see also Ni & Zhou, 2005; Smith et al., 2005). The framework theory approach to conceptual change predicts that overcoming the idea of discreteness is gradual and that, in the process, certain intermediate states of understanding are bound to appear in students. These emerge as students enrich their knowledge with new information about rational numbers and create a bridge between the initial perspective of number and the intended one that is not yet available to the student. Indeed, Vamvakoussi and Vosniadou (in press) found

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that, in addition to students who answered that there are infinitely many numbers in any interval and those who invariably answered that there is a finite number of intermediates, there were students who answered that there are infinitely many intermediates in some, but not in all cases. These students, however, did not just provide answers randomly. Specifically, there was a large effect of the type of the interval end points (i.e. natural numbers, decimals, or fractions) on students' judgments about the number, as well as about the type of the intermediate numbers. For example, many students were more apt to accept that there can be infinitely many numbers, in various forms, between two natural numbers, than between two decimals or two fractions. These findings were in line with the predictions of the framework theory approach to conceptual change and thus provided support for this theoretical position.

Because the framework theory approach makes a rather general claim about the processes of learning counter-intuitive concepts in science and mathematics (Vosniadou et al., 2008), a replication of the Vamvakoussi and Vosniadou study with a different population has a certain theoretical interest. In the event that different populations do not appear to face similar difficulties with respect to the dense ordering of rational numbers, or do not show comparable intermediate states, one could argue that the results of Vamvakoussi and Vosniadou are merely a product of instruction in the context of a particular educational system. Similarly, if students from other populations were found to either succeed or fail across all number types, or make random mistakes, then not only the generality of Vamvakoussi and Vosniadou's findings but also the assumptions of the framework theory of conceptual change would become questionable.

Using the method and instrument reported in Vamvakoussi and Vosniadou (in press), we conducted a cross-cultural comparison study with a Greek and Flemish sample of 9^{th} graders. Belgium and Greece participated in the Programme for International Student Assessment (PISA) for 9th graders in 2000, 2003, and 2006. Greek students were consistently ranked below the OECD average in mathematical tasks, whereas Flemish students were consistently ranked among the best performing students worldwide. Thus, one could expect that a representative sample of Flemish students would perform better than the Greek ones in the tasks used by Vamvakoussi and Vosniadou. From the perspective of the framework theory approach to conceptual change, one would nevertheless expect that Flemish students would be found at similar intermediate levels of understanding of the infinity of numbers in an interval, reflecting comparable difficulties with their Greek peers.

METHOD

Participants

The participants of this study were 84 Greek and 111 Flemish 9^{th} graders.

We chose 9^{th} graders as an interesting age group, because according to the curricula in both countries, they have in principle all knowledge necessary to deal with the research tasks correctly. Starting from the 3rd grade, they have been exposed to systematic instruction about non-natural numbers in the form of decimals and fractions, including ordering, operations, conversions from decimal to fractional form and vice versa, equivalence of fractions, and to decimals with infinite number of decimal digits. By 9th grade, they have been introduced to negative and irrational numbers, and have been taught the terms "rational" and "real" numbers.

The sample was representative of the student population in the two countries: All Greek students came from the most prevalent type of public schools (General Gymnasia) where all students follow the same study stream, and Flemish students were more or less representatively selected from the various study streams they could follow.

Materials

The items we used came from the questionnaire developed by Vamvakoussi and Vosniadou (in press), which was translated first from Greek to English, and then to Dutch. These were multiple-choice items that presented students with pairs of pseudo-successive numbers, such as .005 and .006, or 1/3 and 2/3, asking how many numbers there are in between. The items – that were offered in a randomized order – belonged to 3 item blocks, based on the type of the interval endpoints, namely two natural numbers (Nn, 2 items), two decimals (Dec, 4 items), and two fractions (Fra, 4 items).The answering alternatives offered were the same across all items and were as follows: a) There is no other number, b) there is a finite number of decimals, c) there is a finite number of fractions, d) there are infinitely many decimals, e) there are infinitely many fractions, f) there are infinitely many numbers and they can have different forms, e.g. decimals, fractions, recurring decimals, etc., g) I don't agree with any of the above. I believe that …

Alternative (a), coded as Fin_0, corresponds to the most "naïve" answer, reflecting the idea that the given numbers are successive. Choosing the alternatives (b) or (c), coded as $Fin_{\neq 0}$/specific type, means that the given numbers are not deemed successive, but that the intermediate numbers are still finite in number and also of a specific type, i.e. either decimals or fractions. Alternatives (d) and (e), coded as Inf-/ specific type, are more advanced than the precedent ones in that they allow for infinitely many intermediate numbers; these, however, are again of the same type as the interval endpoints. Alternative (f) is the most sophisticated answer, coded as Inf: The intermediate numbers are infinitely many and may also be of different types. Finally, alternative (g) offered students the option to give an answer different from the ones already presented. We note in advance that very few students made use of this opportunity.

Procedure

The questionnaires were administered during regular school hours in the classrooms. One of the researchers read aloud some clarifications/instructions, which were also written on the first page of the questionnaire. These explained that the term ‘numbers’ referred to real numbers, what it means for a number to be between two other numbers and encouraged students to express a different opinion than the ones offered, by asserting that the latter were provided by students of the same age.

The students had 45 minutes available to complete the questionnaire.

RESULTS

Students’ total mean performance in the 10 items was computed by scoring the Fin_0 answer as 1, the $Fin_{\neq 0}$ as 2, the Inf- as 3, and the Inf as 4. The total mean performance (Greek: M = 2.79, SD = .82, Flemish: M = 3.05, SD =.83 on a maximum of 4) was subjected to an analysis of variance with nationality as between-subjects factor. The results showed a significant albeit weak main effect for nationality, $F(1, 211) = 5.050$, $p < .05$, $\eta^2 = .023$.

Students’ mean performance in Nn, Dec, and Fra were subjected to a repeated measures analysis of variance with number type as within- and nationality as between-subjects factor. The results showed main effects for number type $F(2, 420) = 27.671$, $p < .001$, $\eta^2 = .101$, and nationality, $F(1, 210) = 4.727$, $p < .05$, $\eta^2 = .022$, and no interaction between these two. Figure 1 shows that Greek and Flemish students’ mean performances across the different number types followed a very similar pattern: Students’ mean performance was much better in the Nn items, as compared to the other number types – this difference was found significant (all p’s < .001) in pair-wise comparisons. The mean performance was lower in Dec items, and was lowest for Fra, although there were no significant differences between the latter two types of items. On the other hand, pair-wise comparisons showed that Flemish students performed significantly better (at the .05 level) than the Greek ones across all types of numbers, with the exception of Nn.

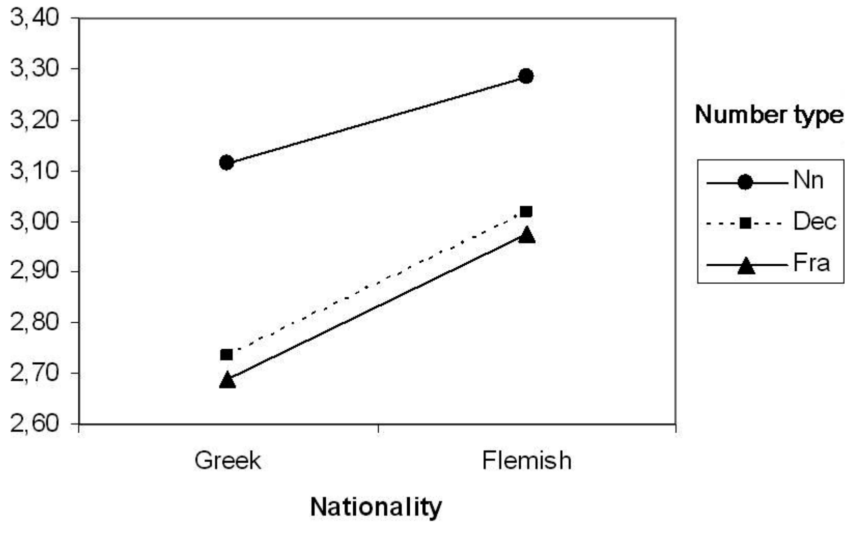

Figure 1: Estimated marginal means of performance in each number type by nationality.

To obtain individual student profiles, we conducted a cluster analysis. We rescored our data so that Fin_0 and $Fin_{\neq 0}$ were collapsed in one response- type, namely the FIN one ("there is a finite number of numbers in the interval, either 0 or ≠0). This is because both responses reflect the idea of discreteness. The FIN response type was scored as 1; consequently, the Inf- and Inf response types were scored as 2 and 3, respectively, leading to a maximum mean performance of 3.

We conducted a two-step cluster analysis on the type of responses given to each of the 10 items, the mean performance within number type, and also on the total mean performance. The analysis yielded a four cluster solution. Thirty-six students (17.0%) were found in the first cluster, hereafter *Finiteness*; 56 (26.4%) in the second, hereafter *Advanced Finiteness*; 58 (27.4%) in the third, hereafter *Naïve Infinity*; and 62 (29.3%) in the fourth, hereafter *Sophisticated Infinity*.

As can be observed in Figure 2, students placed in *Finiteness* gave FIN responses across all number types. For students in *Advanced Finiteness*, FIN responses were dominant for decimals and fractions. In the case of natural numbers, however, the dominant response was Inf. The same, i.e. mostly Inf responses for natural numbers, holds for students in *Naïve Infinity*. These students, however, provided mostly Inf- answers in the case of decimals and fractions; in the latter case, there was also a considerable percent of FIN answers. Finally, students in *Sophisticated Infinity* gave Inf responses across all types of numbers.

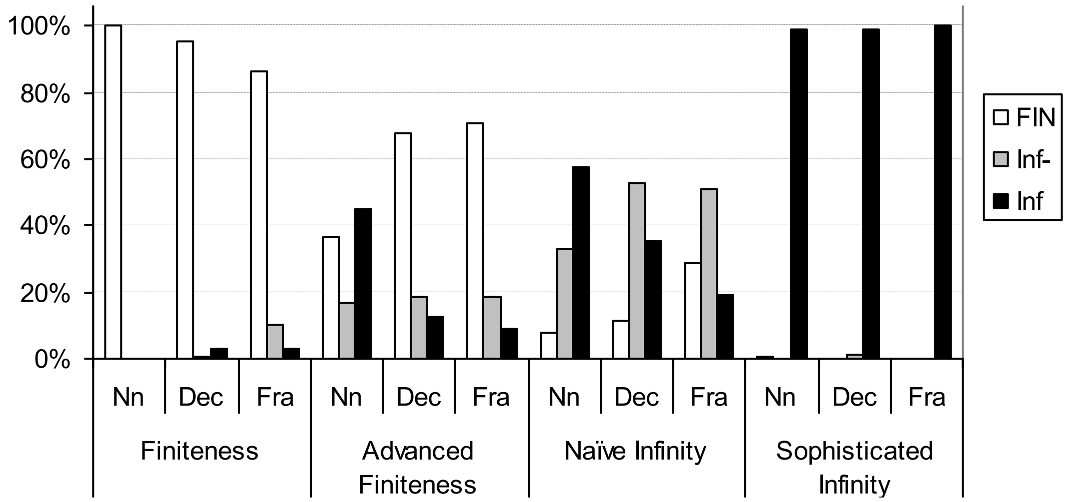

Figure 2: Percent of response types per number type within cluster.

To further substantiate the differences between the clusters, and also within cluster (in particular in the two intermediate clusters, across the different types of numbers), students' mean performances in Nn, Dec, and Fra were subjected to a repeated measures analysis of variance with number type as a within- and cluster as a between subjects factor. The results showed significant main effects for cluster, $F(3, 208) = 802.412$, $p < .001$, $\eta^2 = .920$, and number type, $F(2, 416) = 30.442$, $p < .001$, $\eta^2 = .128$; and significant interaction between cluster and number type, $F(6, 416) = 19.297$, $p < .001$, $\eta^2 = .218$. Post hoc analysis showed that there were significant differences between each cluster and its subsequent one across all type of numbers. Within *Advanced Finiteness*, students performed significantly better (at the .001 level) in Nn, as compared to Dec and Fra. Within *Naïve Infinity*, students performed significantly better at the .001 level in Nn and Dec, as compared to Fra.

Looking at the distribution of Greek and Flemish students across the four clusters (Table 1), one can notice that Greek students were found more often in the first two clusters, while Flemish were found more often in the last two clusters. A chi-square test, however, showed that this difference was not significant, $\chi^2(3, N=212) = 4.870$, $p > .05$.

Nationality	Cluster			
	Finiteness	Advanced Finiteness	Naïve Infinity	Sophisticated Infinity
Greek	18 (21.4%)	23 (27.4%)	25 (29.8%)	18 (21.4%)
Flemish	18 (14.1%)	33 (25.8%)	33 (25.8%)	44 (34.4%)

Table 1: Frequency and percent of students in each cluster, by nationality.

DISCUSSION

As expected, the Flemish students outperformed the Greek ones. Despite the performance differences, however, a salient similarity between Greek and Flemish was that they performed better when the interval endpoints were natural numbers, compared to decimals and also to fractions; the latter tasks appeared to be the most difficult for students. Moreover, the fact that the Flemish students performed better than the Greek ones does not imply that the tasks were trivial for them. Indeed, only 34.4% answered consistently that there are infinitely many intermediates, of different forms, across all tasks; and 14.1% were found consistently on the “finite side”.

More importantly, although more Flemish students were found in *Sophisticated Infinity*, a comparable – and moreover considerable – percent of Flemish and Greek students were placed at intermediate levels of understanding the infinity of numbers in an interval. As expected, students’ judgments at these levels were largely affected by the type of the interval endpoints. Students in *Advanced Finiteness* were on the “finite side” with respect to non-natural numbers, but were more apt to accept the infinity of numbers in the case of natural numbers. Students in *Naïve Infinity* were on the “infinity side” for natural numbers and also decimals. Nevertheless, the idea of discreteness was still present in the case of fractions; in addition, these students were still reluctant to accept that there can be *decimals* between *fractions* and vice versa.

These results are in line with the findings of the Vamvakoussi & Vosniadou (in press) study. In a more general fashion, they are in line with the predictions of the framework theory approach to conceptual change and thus provide support for this theoretical position.

On the other hand, these results raise some interesting questions about rational number instruction in both countries. The Flemish students – coming from an educational system that creates the opportunity for them to be ranked among the best performing students worldwide – appear to face the same conceptual difficulties with the rational number concept as the Greek students. Indeed, our findings indicate that both Greek and Flemish 9th graders, to a large extent, have not overcome the idea that the rational numbers are discrete, like the natural numbers. Moreover, it appears that they are far from conceptualizing rational numbers as a unified system of natural and non-natural numbers, invariant under different symbolic representations. Indeed, if students understood that natural and non-natural numbers are members of the same family, and that (rational) decimals and fractions are interchangeable representations, rather than different kinds of numbers, then they should treat them the same with respect to ordering. Therefore, the similarities, rather than the differences, in mathematics instruction between the two countries are of interest. Such a similarity can be found in the mathematics curricula, at least with respect to the concept of number.

As Greer and Verschaffel (2007) point out, within the research community there is a conceptual shift away from the view of mathematics as cumulative and mathematics learning as enrichment, merely building on prior knowledge without the need for drastic restructuring; this shift, however, has not found its way to the classrooms. Both the Greek and the Flemish curriculum seem to miss the almost trivial – albeit important – fact that the shift from natural to rational number entails qualitative, rather than merely quantitative, changes in the concept of number – in other words: that there is more to rational numbers than "more numbers". Thus Greek and Flemish students alike spend the first years of instruction focusing on natural number arithmetic, validating and strengthening the conception of number as natural number. Then they are gradually introduced to non-natural numbers in the form of fractions and decimals and spend much of their time at elementary school getting familiar with mostly procedural aspects of these new constructs. In the textbooks, the *similarities* between natural and non-natural numbers are strongly emphasized, with a view to making the latter more accessible to students. Then, at the first years of secondary school, the term "rational numbers" is introduced, with the implicit assumption that this knowledge can be built on students' prior experience with natural and non-natural numbers, which will then be integrated in a unified whole. Clearly, this expectation is not met.

In the current literature, various proposals and attempts are made with a view to make the shift from natural to rational numbers less thorny for students (Moss, 2005; Ni & Zhou, 2005). An essential step in this direction is arguably to inform the curricula and textbook designers, as well as the teachers, about the specific difficulties that students are likely to meet in the development of the number concept. We suggest that taking a conceptual change perspective might be valuable in this respect.

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