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BRIDGING THE GAP BETWEEN DISCRETENESS AND DENSITY

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We conducted a short intervention study using expository texts with the purpose of fostering secondary students' understanding of the density of numbers. The participants were 46 8th and 52 10th graders, who were administered a pre-test, then an expository text, and last a post-test. The experimental group was exposed to information about the infinity of numbers in an interval, and also to a 'bridging analogy' between students' initial conceptions of the segment, and the segment as a dense array of points. The experimental group outperformed the control group, who was exposed only to information about the infinity of numbers, and also provided better explanations of their answers.

INTRODUCTION

In this paper we report a short intervention study that investigated the potential instructional value of the number line in secondary school students' understanding of the density of numbers.

The property of density of rational and real numbers is described as the possibility to always find at least one number between any two numbers. This implies that, within a dense set of numbers, between any two elements there are infinitely many others and that no element has a (unique) successor. It is amply documented that understanding density is difficult for students at various levels of instruction (e.g. Lehtinen, Merenluoto, & Kasanen, 1997; Neumann, 1998; Tirosh, et al., 1999). In line with prior research, in previous work (Vamvakoussi & Vosniadou, 2004, 2007) we found that secondary students answered frequently that there is a finite number of numbers in intervals defined by rational numbers. In addition, our findings showed that the symbolic representation of the numbers defining the interval had a significant effect on students' responses. More specifically, students were very reluctant to accept that there can be decimals between fractions and vice versa; they also treated integers, decimals and fractions differently with respect to the number of intermediate numbers, for instance a student might answer that there are infinitely many intermediates between decimals, but a finite number of intermediates between fractions. These findings suggest that, in this context, students treated different symbolic representations of numbers as if they were different numbers, indicating a view of the rational numbers set as consisting of different, unrelated 'sets' (i.e. integers, decimals, fractions).

The number line is grounded on the analogy "numbers are points on the line". As such, it calls for a re-conceptualization for numbers, which might arguably help students conceive of rational numbers as individual entities, and also facilitate their

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understanding that, for instance, 0.5 and 1/2 are interchangeable representations of the same number, rather than different numbers, since they correspond to the same point. This could promote students' understanding of rational (and real) numbers as a unified number system (Kilpatrick et al., 2001). In addition, being a continuous representation itself, the number line could (in principle) be used to confront students' belief that numbers are discrete in nature. From a historical point of view, various descriptions of density emerged in mathematicians' attempts to capture the characteristic properties of the geometrical line, far before the emergence of any notion of the arithmetic continuum (e.g. Bell, 2005; Klein, 1968). In a pilot study (Vamvakoussi & Chatzimanolis, 2008) we found that secondary school students (grades 7^{th} -11^{th}) were more apt to accept the infinity of points on a segment, than the infinity of intermediate numbers in an interval.

However, the number line is a highly abstract analog itself, which requires from students to coordinate their understandings coming of two different domains, namely the domain of number and that of geometrical magnitudes, and in addition introduces a number of new conventions to be learned (English, 1993).

Lakoff and Núñez (2000), make a sharp distinction between the notion of the 'holistic line', which can be conceptualized as the trace of a moving object (e.g. the pencil, when it does not leave the paper) and is continuous in an everyday sense, and the notion of the line as a set of points, which they characterize as a mathematically elaborated metaphor of the line. They argue that these are two conflicting images of the line, and that the fact that students are not aware of this conflict contributes to their difficulty in interpreting the number line. Moreover, they argue that the number-points correspondence is far from being transparent to students.

Dealing with the geometrical line per se presents students with difficulties, since they may not distinguish between the abstract, geometrical line and its physical representations, such as a line drawn by a pencil on a piece of paper bearing features such as width, which is not supposed to belong to its idealized counterpart (Fischbein, 1987). Such considerations may also interfere with students' understanding about the number of points of a segment. More specifically, conceptions of points as material spots are at odds with the infinite amount of points on a line segment. Such conceptions may underlie students' belief that longer segments have more points (Fischbein, 1987). Reviewing the related literature, Sbaragli (2006) points to the conception of a segment as "a necklace of beads", put the one immediately next to the other, which seems to underlie many secondary students' descriptions of the structure of the segment.

Students' early experiences with the number line at the elementary school should also be taken into consideration. Quoting Dufour-Janvier, Bednarz, and Belanger (1987), English (1993) points out that students tend to see the number line as a series of 'stepping stones' with an empty space in between, commenting that *"this may explain why so many secondary students say that there is no numbers, or at the most* *one, between two whole numbers"* (p. 24). Another common metaphor for the number line, namely that of the ruler (e.g. Doritou & Gray, 2007) may also convey the idea that the number of numbers in an interval is finite.

Findings from our pilot study were compatible with the above considerations. More specifically, our participants described the geometrical line as a real-world object that gets thicker if magnified; or, given the possibility of unlimited magnification, they described the segment as consisting of points, the one immediately next to the other. Interestingly, the latter description was offered also by students who had consistently answered that there are infinitely many points on a segment. We took this to be an indication that, from the student' point of view, the 'infinity many intermediates' aspect of density does not necessarily imply its 'no successor' aspect. This assumption is in line with findings showing that students believe that, for instance, 2.9999… is immediately before 3, i.e. such number in principle exists, albeit it can not be precisely defined (e.g. Lehtinen et al., 1997).

The question rises, how can the gap between the segment as conceived by students and the segment as a dense array of points be bridged? We drew on the 'bridging analogy' approach, developed by Clement and his colleagues (see, for example, Clement, 1993). This approach involves the interpolation, between students' initial understanding of a situation and the intended scientific idea, of one or more intermediate anchoring situations, expected to trigger a correct intuition, i.e. one that can be developed toward understanding the target situation. We devised the "rubber line" anchor: "The line is like an imaginary rubber band that never breaks, no matter how much it may be stretched". This analogy is (partially) grounded on students' experience with a real world object and it aims at conveying the idea that no matter how close two points seem to be, there are always more points to be found in between, by stretching the rubber line. The 'rubber line' is compatible with students' conceptions of the geometrical line, which lead them to believe that there will eventually be two successive points, but explicitly contradicts this expectation.

We designed a short, text -based intervention with the purpose of investigating the added value of this approach in students' understanding of the denseness of points and numbers on a segment and in an interval, respectively. We assumed that students exposed to explicit information about the infinity of numbers in an interval would improve their performance in similar tasks. However, we hypothesized that students exposed to the 'rubber line' analogy would perform better in items related to the 'no successor' aspect of density and would provide better explanations for their answers.

METHOD

Participants.

The participants were 46 8^{th} and 52 10^{th} graders, from 4 classes of the same school in the suburbs of Athens.

Materials.

We constructed two texts (T_{INF}, T_{RL}). T_{INF} reminded students that all numbers can be placed on the number line. Then it referred to 0 and 1 on the number line and evoked the notion of 'space' between them. It provided the correct answer ("there are infinitely many numbers between 0 and 1"), accompanied with several examples. The first part of T_{RL} was identical to T_{INF}. In its second part, the 'rubber line' anchor was employed to explain how it is possible for two points to 'look' as if they were successive, and yet have infinitely many points in between. T_{RL} concluded by emphasizing the numbers-points correspondence and the implication that there are infinitely many numbers in any interval.

We designed two questionnaires as pre- and post-tests. They had 9 forced-choice items in common, focusing on the infinity of numbers in an interval and the infinity of points on a straight segment ('infinity items'). The post-test included 5 additional items, asking students to evaluate a statement about the existence of two successive numbers, and to justify their answer ('no successor' items).

Procedure.

The students were administered a) the pre-test, b) the expository text, and c) the post-test. Students in the same class received the same type of text. The procedure lasted 45 minutes.

RESULTS

Students' mean performance in the common items of the pre- and the post-test was computed by scoring the "Finite number" answer as 1 and the "Infinitely many" answer as 2 (see Table 1).

	Text type	N	Pre-test		Post-test	
			Mean	S.D.	Mean	S.D.
8th grade	T_{INF}	25	1.227	.188	1.640	.291
	T_{RL}	21	1.302	.298	1.651	.332
10th grade	T_{INF}	25	1.351	.337	1.782	.237
	T_{RL}	27	1.391	.394	1.848	.239

Table 1: Mean scores in the pre- and post-test, as a function of grade and text-type.

No significant performance differences between the two age groups, or between the two text type conditions within grade, were found before the intervention (tested with Mann-Whitney U test).

Students' performance in the 'infinity' items increased after the intervention (Table 1). A Wilcoxon signed ranks test, comparing students' mean performance between the pre- and the post-test, within grade and text-type condition, showed that this difference was significant, for all groups. More specifically, 8th graders' performed significantly better under the T_{INF} condition, z= -3.920, p<.001, and also the T_{RL} condition, z=-3.861, p<.001. Similarly for 10th graders, under the T_{INF} condition, z= -3.374, p<.001, and the T_{RL} condition, z=-3.895, p<.0001.

No significant differences in students' performance in these items' were found between the two text-type conditions, for any of the grades (tested with Mann-Whitney U test).

Finally, a Mann-Whitney U test comparing 8th and 10th graders' performance in the post-test showed a significant difference, z=-3.756, p<.001, in favour of 10th graders.

Students' responses in the additional, 'no successor' items of the post-test were categorized as incorrect, partially correct and correct and were scored as 1, 2, and 3, respectively, by two independent scorers (mean scores are presented in Table 2). For a student to be credited with a correct response, she had to make a correct choice, i.e. deny the possibility of the two given numbers (or points) to be successive and also present a principle-based explanation. A response was categorized as 'partially correct' when the student made the correct choice, but offered no explanation (most typical case), or came up with a counter-example without referring to a more general principle. So for example, a correct choice accompanied with the explanation "*It is not necessary for 2.002 to be immediately after 2.001. Take for example 2.0015 or 2.0012.*" was scored by 2, whereas a correct choice accompanied with the explanation "*Between 3/7 and 4/7 there are infinitely many numbers, so we can always find a number closer to 3/7*" was scored by 3.

	Text type	N	'No successor' items	
			Mean	S.D.
8th grade	T_{INF}	25	1.544	.743
	T_{RL}	21	1.968	.852
10th grade	T_{INF}	25	2.088	.698
	T_{RL}	27	2.467	.716

Table 2: Mean scores in the additional items of the post-test.

A Mann-Whitney U test comparing students' mean performance between the T_{INF} and T_{RL} conditions, within grade, showed significant difference in the case of 10th graders, z= -2.159, p<.05.

The added value of the T_{RL} became clearer when we looked at the number of incorrect responses in the 'successor items' of the post-test. As can be seen in Table 3, students not exposed to the 'rubber line' text were far more often found to deem at least one pair of numbers (or points) successive, than the T_{RL} students. It is interesting to note that, in this respect, the T_{RL} 8th graders outperformed the older students who were not exposed to the rubber-line text.

Grade	Incorrect answers in the 'successor items'	T_{INF}	T_{RL}	Total
8th	None	6 (24%)	12 (57.1%)	18 (100%)
	At least one	19 (76%)	9 (42.9%)	28 (100%)
	Total	25 (100%)	21(100%)	46 (100%)
10th	None	8 (32%)	18 (66.7%)	26 (100%)
	At least one	17 (68%)	9 (33.3%)	26 (100%)
	Total	25 (100%)	27 (100%)	52 (100%)

Table 3: Frequencies and percents of students who did or did not make a mistake in the 'no successor items', as a function of grade and text-type.

Table 4 presents a categorization of students based on the number of correct answers they gave in the 5 'no successor' items. It can be noticed that students exposed to the 'rubber line' text provided more often elaborated explanations than their T_{INF} fellow students, within both age groups.

		Number of correct answers in the 5 'no successor' items			
Grade	Text type	0 or 1	2 or 3	4 or 5	Total
8th	T_{INF}	17 (68%)	5 (20%)	3 (12%)	25 (100%)
	T_{RL}	9 (42.9%)	5 (23.8%)	7 (33.3%)	21 (100%)
	Total	37 (56.5%)	15 (22.7%)	14 (21.2%)	46 (100%)
10th	T_{INF}	12 (48%)	3 (12%)	10 (40%)	25 (100%)
	T_{RL}	6 (22.2%)	5 (18.5%)	16 (59.2%)	27 (100%)
	Total	25 (33.8%)	18 (24.3%)	31 (41.8%)	52 (100%)

Table 4: Categorization of students based on the number of correct answers in the 'no successor' items of the post-test, as a function of grade and text-type.

We note that students exposed to the 'rubber line' text employed this information, and the numbers-points correspondence in general, in justifying their answers (see Examples 1 and 2).

Example 1. (Two points cannot be found the one immediately next to the other) *"Because you keep stretching the line and you find that there are more points - this process does not end."*

Example 2. (2.002 cannot be the successor of 2.001) "*Because if you place them on the number line, there are infinitely many points between these numbers, and therefore infinitely many other numbers*".

CONCLUSIONS- DISCUSSION

Our findings showed that providing information via an expository text about the infinity of numbers in a specific interval (i.e. the one defined by 0 and 1) helped students improve their performance in similar tasks, under all conditions. Tenth graders profited in general more from this short intervention than the younger students, since their performance did not differ significantly before the intervention, but they performed significantly better afterwards. This finding can be attributed to the fact that older students were more apt to extract information provided by the texts and connect it to their existing knowledge, than the younger ones.

The added value of the 'rubber line' anchor manifested in students' responses in the additional items of the post-test, which focused on the 'no successor' aspect of density. Based on findings from our pilot study (Vamvakoussi & Chatzimanolis, 2008), we took this aspect to be more difficult for students, than the 'infinitely many intermediates' aspect of density and we addressed it in geometrical context, bridging students' conceptions of the segment and its relation with points and the notion of a segment as a dense array of points via the 'rubber line' anchor. We also emphasized the numbers-points correspondence, with the purpose to facilitate students to transfer the information about the denseness of points to the domain of numbers. Our results showed that both 8^{th} and 10^{th} graders profited of this approach, since they were more consistent in denying the possibility of two numbers or points to be successive, and were also more apt to provide explanations, than their fellow students who were exposed only to information about the infinity of numbers in an interval.

Keeping in mind that, besides being short, the intervention was rather conservative in the sense that it was text-based and did not allow for any interaction in the classroom, our findings suggest that purposeful, long-term use of the number line in instruction maybe valuable, provided that it is accompanied with adequate explanations that bridge the gap between students' conceptions of the number line and the intended mathematical meanings.

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