proceedings of the 4th mediterranean conference on mathematics education

MEDCONF 2005

Palermo – Italy
28-30 January 2005

Editors:
A. Gagatsis, F. Spagnolo, Gr. Makrides, V. Farmaki

Organizers:
University of Palermo
Cyprus Mathematical Society
MATHEU Project (Socrates Comenius)

Volume II

CONTENTS

VOLUME II

WHAT CAN WE GAIN FROM A CONCEPTUAL CHANGE APPROACH TO THE LEARNING AND TEACHING OF MATHEMATICS?

Stella Vosniadou, Xenia Vamvakoussi and Konstantinos P.Christou
Cognitive Science Laboratory, Department of Philosophy and History of Science (MITHE)
University of Athens, Panepistimioupolis, A. Ilisia, Athens, Greece
svosniad@phs.uoa.gr, xenva@phs.uoa.gr, kochrist@phs.uoa.gr

ABSTRACT

The term conceptual change is used to characterize the kind of learning required when the new information to be learned comes in conflict with the learners' prior knowledge. The conceptual change approach has been applied extensively to explain students' difficulties in science learning. In this paper, we argue that the conceptual change approach can be fruitfully applied to mathematics learning and teaching. We discuss the results of three empirical studies that demonstrate the usefulness of the conceptual change approach in explaining students' difficulties in the areas of fraction, the dense structure of rational numbers, and the use of literal symbols in algebra. The instructional implications of these studies are also discussed.

INTRODUCTION

The term *conceptual change* is used to characterize the kind of learning required when the new information to be learned comes in conflict with the learners' prior knowledge usually acquired on the basis of everyday experiences. It is claimed that in these situations, a major reorganization of prior knowledge is required – a conceptual change. Some of the situations were conceptual change is required involve, for example, the acquisition of the scientific concept of force which comes in conflict with the everyday concept of force as a property of physical objects (Chi, Slotta, & de Leeuw, 1994), understanding the Copernican view of the solar system which comes in conflict with the geocentric view (Vosniadou & Brewer, 1994), and the acquisition of the concept of fraction as it requires radical changes in the pre-existing concept of natural number (Hartnett & Gelman, 1998; Stafylidou & Vosniadou, 2004).

Some researchers in learning and instruction (e.g., Caravita & Halden, 1994) ask: Why should we call this type of learning 'conceptual change' and not just 'learning'? While conceptual change is undeniably a form of learning, it is important to differentiate conceptual change from other kinds of learning because it requires different mechanisms to be accomplished and different instructional interventions to be taught. Most learning is additive and involves an enrichment of existing knowledge. Conceptual change cannot, however, be achieved through additive mechanisms. In fact, as we will argue later in greater detail, the use of additive mechanisms in situations requiring conceptual change is one of the major causes of misconceptions.

A common type of misconception is caused when new information is added to an incompatible knowledge base, producing synthetic models, like 'the hollow sphere' (Vosniadou & Brewer, 1992), or the belief that fractions are always smaller than the unit (Stafylidou & Vosniadou, 2004). It is important in instruction to distinguish cases requiring conceptual change and alert students against the use of additive mechanisms in these cases. More generally, it is stressed that it is important to develop intentional learners that have acquired the metacognitive skills to correctly identify different kinds of learning and apply the most effective strategies in dealing with them (Vosniadou, 2002).

DIFFERENT APPROACHES TO CONCEPTUAL CHANGE

The conceptual change approach was brought to the field of learning and instruction from the philosophy and history of science (Kuhn, 1970; Lakatos, 1970) by science educators who saw certain analogies between theory changes in the history of science and students' learning of science (e.g., Posner, Strike, Hewson, & Gertzog, 1982). Since the 1970's researchers such as Viennot (1979), Novak (1977) and Driver and Easley (1978) realized that students bring to the science learning task alternative frameworks or misconceptions that are robust and difficult to extinguish. Posner et al. (1982, and also McCloskey, 1983) saw these alternative frameworks as theories that need to be replaced by the currently accepted, correct scientific views through a process of conceptual change. Drawing on Kuhn (1970), Posner et al. (1982) argued that in order for students to be able to replace their alternative conceptual frameworks with the currently accepted scientific views: a) there must be dissatisfaction with existing conceptions, b) the new conception must be intelligible, c) the new conception must appear initially plausible, and d) the new concept should suggest the possibility of a fruitful program.

The Posner et al. (1982) theoretical framework became the leading paradigm that guided research and instructional practices in science education, until it became subject to several criticisms (i.e., Caravita & Halden, 1994; Smith, diSessa, & Rochelle, 1993). These criticisms pointed out that the conceptual change approach focuses on the mistaken qualities of students' prior knowledge and ignores their productive ideas, that alternative conceptions may be not as robust as they seem to be, that cognitive conflict is not an effective instructional strategy, and that instruction that "confronts misconceptions with a view to replacing them is misguided and unlikely to succeed" (Smith et al., p. 153). Caravita and Halden (1994) also pointed out that conceptual change happens in a larger situational, educational, and socio/cultural context, that it is affected by motivational and affective variables, and that we need to recognize that science is socially constructed and validated (see also Driver, Asoko, Leach, & Mortiner, 1994; and Pintrich, 1999).

We agree with all of the above-mentioned criticisms of the original conceptual change approach. Most important, we find a great deal of truth in the recommendation to study the knowledge acquisition process in greater detail, and in particular, the need to focus on "detailed descriptions of the evolution of knowledge systems" (Smith et al., 1993, p. 154) over long periods of time. Indeed, research in cognitive development provides an important source of information about the processes of conceptual change (e.g., Carey, 1985; Gallistel & Gelman, 1992; Hatano & Inagaki, 1998). Vosniadou and her colleagues have attempted to provide a cognitive developmental approach to conceptual change through detailed descriptions of the development of knowledge in several areas of the natural sciences, such as observational astronomy (Vosniadou & Brewer, 1992; 1994; Vosniadou, 1994; 2003) mechanics (Ioannides & Vosniadou, 2001; Megalakaki, Ioannides, Vosniadou & Tiberghien, 1997), geophysics (Ioannidou & Vosniadou, 2001), chemistry (Kouka, Vosniadou, & Tsaparlis, 2001), and biology (Kyrkos & Vosniadou, 1997). The results of these studies have shown that young children answer questions about force, matter, the earth in space, or about the composition of earth, mostly in an internally consistent way, revealing the existence of narrow but coherent initial explanatory frameworks. These explanatory frameworks are different in their structure, in the phenomena they explain, and in their individual concepts, from the scientific theories to which children are exposed through systematic instruction. The process of learning science is a slow and gradual one, during which children usually add the new, scientific, information to their initial explanatory frameworks, destroying their coherence and creating synthetic models. Examples of such synthetic models are the model of the dual sphere, the hollow sphere, or the flattened sphere, the model of the sun and the moon revolving around a spherical earth in a geocentric solar system, etc. (see Vosniadou & Brewer, 1992; 1994).

One could argue that the cognitive developmental approach to conceptual change is not very different from the classical conceptual change approach put forward by Posner et al. (1983). But this is not the case. The cognitive developmental approach to conceptual change meets all the criticisms of Smith et al. (1993). First, misconceptions are not considered as unitary, faulty conceptions that represent a different physical theory. Rather, we describe a knowledge system consisting of many different elements organized in complex ways. Second, we make a distinction between the learner's initial explanatory framework, prior to systematic instruction, and misconceptions that are produced after instruction. We believe that most of these misconceptions can be characterized as synthetic models – i.e., attempts by learners to synthesize the new information with the initial explanatory framework. Third, our theoretical position is a constructivist one. Not only it assumes that new information is built on existing knowledge structures; it also uses constructivism to explain students' misconceptions and to provide a comprehensive framework for making meaningful and detailed predictions about the knowledge acquisition process. Finally, while our cognitive approach investigates only one facet of conceptual change, it is complementary and not contradictory to other approaches that deal with motivational/affective and socio/cultural factors (Anderson, Greeno, Reder, & Simon, 2000)

CONCEPTUAL CHANGE AND MATHEMATICS LEARNING

Although some historians of mathematics find the Kuhnian conceptual change approach particularly fruitful in the case of mathematics (e.g., Corry, 1993; Dauben, 1984; Kitcher, 1983), there is a general reluctance in the philosophy and history of mathematics circles to apply the conceptual change approach to mathematics. That mathematics is based on deductive proof and not on experiment, that it is proven to be extremely tolerant of anomalies, and it does not display the radical incommensurability of theory before and after revolution, are some of the reasons why Thomas Kuhn himself exempted mathematics from the pattern of scientific development and change presented in the Scientific Revolutions (see Mahoney, 1997). According to Mahoney (1997) "synthetic geometry, invariant theory, or quaternions may lose interest for mathematicians, subjects may be judged obsolescent or fruitless, but they do not seem to cease to be mathematics in the way that Aristotle's mechanics ceased to be mechanics, or Galen's physiology ceased to be physiology, or phlogiston chemistry ceased to be chemistry"[1] (p. 2). Unlike science, the formulation of a new theory in mathematics usually carries mathematics to a more general level of analysis and enables a wider perspective that makes possible solutions that have been impossible to formulate before (Corry, 1993; Dauben, 1984).

We find this discussion particularly interesting because it helps to illuminate some of the debates that have taken place in the conceptual change approach literature as it applies to learning situations. More specifically, a number of researchers have pointed out that even in the case of the natural sciences conceptual change should not be seen in terms of the replacement of students' naïve physics with the 'correct' scientific theory but in terms of enabling students to develop multiple perspectives and/or more abstract explanatory frameworks with greater generality and power (Driver et al., 1994; Spada, 1994). It thus appears that the theory replacement issue (which represents a significant difference in the historical development of the natural sciences compared to mathematics) may not be an issue in the case of learning and instruction. In fact, students are confronted with similar situations when they learn both mathematics and science. As it is the case that students develop a naïve physics on the basis of everyday experience,

---

[1] The common explanation of this difference is that "mathematicians deal with worlds of their own making, which their encounters with the physical world cannot constrain them to change" (Mahoney, 1997, p. 2).

they also develop a 'naïve mathematics', which appears to be neurologically based (developed through a long process of evolution), and consists of certain core principles or presuppositions (such as the presupposition of discreteness in the number concept) that facilitate some kinds of learning but inhibit others (Dehaene, 1998; Gelman, 2000; Lipton & Spelke, 2003). Similarities such as the above, support the argument that the conceptual change approach can be fruitfully applied in the case of learning mathematics. In the section that follows we describe three studies conducted in our lab that explore the conceptual change approach for mathematics learning.

THREE STUDIES FROM A CONCEPTUAL CHANGE PERSPECTIVE

Already at a very young age children have formed what has been characterized as 'principled understanding' of natural numbers (Gelman, 2000), which is based on the act of counting. A basic characteristic of this initial understanding of numbers is that numbers are discrete in nature. In fact, there is evidence that the property of discreteness of numbers may be neuro-biologically based, in the sense that humans are predisposed to learn and reason with natural numbers (Dehaene, 1998; Gelman, 2000). As Greer (2004) argues, an intuition of continuous change in a quantity might also be innate. However, in the early years of a child's life, there are no representational tools available to support the externalization of that intuition.

During the first years of mathematics education, children's informal knowledge about natural numbers is further confirmed and strengthened. There is a great deal of evidence that when new knowledge about rational numbers is to be acquired prior knowledge about natural numbers may inhibit further understandings. These are the cases in which what is to be learned comes in conflict with what is already known. We suggest that in these cases, conceptual change is required. We argue that some of the cases where conceptual change is required are the following: a) in the development of the concept of fraction, b) the density of rational numbers, and c) the use of literal symbols in the case of algebra. More specifically, according to the conceptual change framework we assume that students' understanding about fractions, density, and the use of literal symbols in algebra will be constrained by their initial 'number theories'. Therefore we expect students to generate errors that reflect these presuppositions. We also expect that the development of the mathematical concepts in all three cases will be a slow and gradual process that reflects students' efforts to assimilate new information to their pre-existing knowledge. Therefore, we expect that students who belong to intermediate levels of understanding have misconceptions that can be explained as *synthetic models* (Vosniadou, 1994).

We report three studies that test the predictions of the conceptual change framework in the case of mathematics learning. In the first study, Stafylidou and

Vosniadou (2004a) examined the development of the concept of fraction. They pointed out that fractions differ from natural numbers a) in their symbolic representation (one number versus two cardinal numbers separated by a line), b) in that one cannot use counting-based algorithms for ordering them (fractions do not have unique successors; there are infinitely many numbers between any two fractions), and c) with respect to the unit; while the unit is the smallest natural number, there is no ''smallest ''rational number.

Stafylidou and Vosniadou hypothesized that the presuppositions of discreteness interferes in the acquisition of the concept of fraction, causing systematic misconceptions. It was also hypothesized that most of these misconceptions can be explained as attempts on the part of the children to assimilate information about fractions into the explanatory framework for natural numbers. This hypothesis was tested in a sample of two hundred students ranging in age from 10 to 16, using a questionnaire that required them to decide on the smallest/biggest fraction and to order a set of given fractions.

The results showed that there are indeed systematic misconceptions of fractions that reveal the interference of prior knowledge about natural numbers. More specifically, three main explanatory frameworks for interpreting fractions emerged from the study. The first explanatory framework is that fractions consist of two independent numbers. Children who adopt this framework interpret fractions within the explanatory framework for natural numbers. The children who adopt the second explanatory framework consider fractions as parts of a whole. These children transfer to fractions the presupposition of natural numbers that the unit is the smallest number. Only the students assigned in the third explanatory framework were able to understand the relation between numerator and denominator and to consider that fractions can be smaller, equal, or even bigger than the unit. The results are consistent with the hypothesis that the acquisition of the concept of fraction is a gradual process that proceeds through a number of misconceptions, or synthetic models, that reveal the effects of the explanatory framework for natural numbers. As such, they support the hypothesis that the development of understanding of the numerical value of fractions requires conceptual change.

In a second line of research, Vamvakoussi and Vosniadou (2004b) assumed that the idea of discreteness is a fundamental presupposition of children's initial 'number theories' that constrains their understanding of the dense structure of the set of rational numbers. Analyzing the differences between the set of natural and the set of rational numbers, they assumed that there are two more cognitive constrains that are bound to stand in the way of children's understanding of density. More specifically, they pointed out that:

- in the set of natural numbers, every number has unique symbolic representation, whereas in the set of rational numbers, every number has multiple symbolic representations.
- the set of natural numbers consists of 'homogenous' elements, in the sense that it consists natural numbers only, whereas the elements of the set of

rational numbers are 'heterogeneous', in the sense that it consists of natural and non natural numbers.

Based on these observations they assumed that students' understanding of the structure of the set of rational numbers would also be constrained by a) their belief that different symbolic representations of the same number refer to different numbers, and b) their disposition to group numbers together on the basis of their symbolic representations.

These hypotheses were tested in two studies. In the first, (Vamvakoussi & Vosniadou, 2004a) the participants were 16 ninth graders, who participated in a 45-minute individual interview during which they dealt with density-related tasks, focusing on the number of numbers there are between two rational numbers (integers, decimals, fractions). In the second, 301 students (164 ninth and 137 eleventh graders) were given a forced-choice questionnaire and one with open ended questions (Vamvakoussi & Vosniadou, 2004b).

The results of the studies showed that the idea of discreteness is very strong, both in the case of ninth and eleventh graders, and that giving up the idea of discreteness is not an 'all or nothing situation'. For instance, the knowledge that there are infinitely many numbers between two decimals is not necessarily transferred to the case of fractions. The belief that different symbolic representations of a number refer to different numbers is reflected in the answers of students of various levels of performance in the questionnaire. To use an example coming from the interviews of the pilot study, a 9th grader answered that there is no other number between 3/8 and 5/8 and he explained that "if you simplify 4/8, you get 1/2 and this is not in between." Another ninth grader explained that there are many numbers between these fractions and she mentioned 8/16 and 4.0/8 and other equivalent fractions as examples. Finally, the effect of the number line on students' performance was quite limited and disappeared when the number line was taken away.

The above results are compatible with the predictions coming form the conceptual change theoretical framework and therefore they support our hypothesis, that understanding the dense structure of rational numbers requires reorganization of prior knowledge structures, namely conceptual change.

A third empirical study by Christou and Vosniadou (2004), investigated students' misunderstandings and difficulties in the use of literal symbols as mathematical objects. When it comes to algebra, the concept of generalized number appears. The generalized number is represented by a letter from the alphabet which stands for every numerical value, unless otherwise specified. The literal symbol, for example 'a', has a univocal form and this might affect students to interpret it's use as standing only for natural numbers who also have a univocal form, for example 2. In addition to that, the form of the literal symbol would also affect students to think that different letters would stand for different arithmetical values. According to these remarks and the conceptual change theoretical framework, Christou and Vosniadou (2004) assumed that students would tend to interpret the use of literal symbols in algebra as symbols that stand for natural numbers and not as fractions, decimals or real numbers. More specifically, the hypotheses of this research were that students would tend to assign a) natural numbers instead of real numbers to the literal symbols, and b) numerical values that maintain the form of the symbol. Another hypothesis was that the students will tend to believe that different literal symbols represent different numbers.

These hypotheses were tested in a study of 128 students from two public high schools in Greece, (76 8^{th} graders and 52 6^{th} graders). The students were asked to assign numerical values to the following algebraic objects that contain literal symbols: a, -b, 4b, a/b, a+a+a, k+3. The results showed that the students are greatly affected by the *form* of the algebraic object that contains literal symbols. For example, the greater majority of the students believed that –g is always a negative number and a/b always represents a positive fraction. In addition, when they assign numerical values to the literal symbols students tend to assign *only natural numbers* and very rarely fractions or decimals. However, it was also found that the tendency to assign natural numbers to the literal symbols was stronger than the tendency to maintain the form in both grades in all algebraic objects.

The above-mentioned approach has the potential to explain students' difficulties in understanding and dealing with advanced mathematical concepts such as functions, absolute value, graphs, in fields like analysis, algebra and analytic geometry where literal symbols are used extensively. It is very important for the students to understand the generalized nature of the use of literal symbols mathematics for their future mathematical development. It can also provide new elements about the way students give meaning to the use of literal symbols as mathematical objects.

WHAT DO WE GAIN FROM A CONCEPTUAL CHANGE APPROACH IN MATHEMATICS LEARNING AND TEACHING?

We are not the first to argue that there may be discordances and conflicts between many advanced mathematical concepts and "naïve mathematics." Fischbein (1987) was one of the first mathematical educators to notice that intuitive beliefs may be the cause of students' systematic errors in mathematics, a fact also noted by Stavy and Tirosh (2000) in their intuitive rules theory (see also Tirosh & Tsamir, 2004), by researchers such as Greeno (1991) and Verschaffel and De Corte (1993) in the case of addition and subtraction, and pointed out by many other math educators such as Vergnaud (1990) and Sfard (1987). Other researchers have argued that incompatibility between prior knowledge and incoming information may be the source of students' difficulties in understanding algebra (Kieran, 1992), fractions (Hartnett & Gelman, 1998), rational numbers (Merenluoto & Lehtinen, 2002), etc. The conceptual change approach has the potential to enrich a social constructivist perspective, provide the needed framework to systematize the above-mentioned widespread findings, and utilize them for a theory of mathematics learning and instruction.

Some of the more obvious advantages of exploring the instructional implications of the conceptual change approach are the following: It can be used as a guide to identify concepts in mathematics that are going to cause students great difficulty, to predict and explain students' systematic errors and misconceptions, to provide student-centered explanations of counter-intuitive math concepts, to alert students against the use of additive mechanisms in these cases, to find the appropriate bridging analogies, etc. In a more general fashion, it highlights the importance of developing students who are intentional learners and have developed the metacognitive skills required to overcome the barriers imposed by their prior knowledge (Schoenfeld, 1987; Vosniadou, 2003).

REFERENCES

Anderson, J., Greeno, J.G., Reder, L.M., & Simon, H.A. (2000). Perspectives on learning, thinking, and activity. *Educational Researcher*, *29*(4), 11-13.

Caravita, S., & Hallden, O. (1994). Re-framing the problem of conceptual change. *Learning and Instruction, 4*, 89-111.

Carey, S. (1985). *Conceptual change in childhood*. Cambridge, MA: MIT Press.

Corry, L. (1993). Kuhnian issues, scientific revolutions and the history of mathematics. *Studies in History and Philosophy of Science, 24*, 95-117.

Chi, M.T.H., Slotta, J.D., & de Leeuw, N. (1994). From things to processes: a theory of conceptual change for learning science concepts. *Learning and Instruction, 4*, 27-43.

Christou, K., P. (2004). Conceptual Change in the transition from arithmetic to algebra. In S.Vosniadou, C. Stathopoulou, X. Vamvakoussi, & N. Mamalougos (Eds.), *4th European Symposium on Conceptual Change: Philosophical, Historical Psychological and Educational Approaches* (pp.115-117). May, 19-23, 2004, Delphi, Greece. Athens: Gutenberg Press.

Dauben, J., (1984). Conceptual revolutions and the history of mathematics: two studies in the growth of knowledge. Originally appeared in E. Mendelsohn (Ed.) *Transformation and tradition in the sciences, Essays in Honor of I. Bernard Cohen*, (pp. 81-103). Cambridge University Press. Reprinted in D. Gilies (Ed.) *Revolutions in mathematics* (pp. 15-20). Oxford University Press, 1992.

Dehaene, S. (1998). *The number sense: How the mind creates mathematics.* Harmondsworth Middlesex England: The Penguin Press. First published by Oxford University Press, 1997.

Driver, R., & Easley, J. (1978). Pupils and paradigms: A review of literature related to concept development in adolescent science students. *Studies in Science Education, 5*, 61-84.

Driver, R., Asoko, H., Leach, J., Mortimer, E., & Scott, P. (1994). Constructing scientific knowledge in the classroom. *Educational Researcher, 23*(7), 5-12.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: Kluwer Academic Press.

Gallistel, G.R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44*, 43-74.

Gelman, R. (2000). The epigenesis of mathematical thinking. *Journal of Applied Developmental Psychology, 21*, 27-37.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*, 170-218.

Greer, B. (2004). The growth of mathematics through conceptual restructuring. In S. Vosniadou & L. Verschaffel, *The Conceptual Change Approach to Mathematics Learning and Teaching, Special Issue of Learning and Instruction*, 14(5), 541-548.

Hartnett, P.M., & Gelman, R. (1998). Early Understandings of number: Paths or barriers to the construction of new understandings? *Learning and Instruction, 8*, 341-374.

Hatano, G., & Inagaki, K. (1998). Qualitative changes in intuitive biology. *European Journal of Psychology of Education, 12* , 111-130.

Ioannides, C., & Vosniadou, S. (2001). The changing meanings of force: From coherence to fragmentation. *Cognitive Science Quarterly, 2*(1), 5-62.

Ioannidou, I., & Vosniadou, S. (2001). The development of knowledge about the composition and layering of the earth's interior. *Nea Paedia, 31*, 107-150 [in Greek].

Kieran, C. (1992). The learning and teaching of school algebra. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kouka, A., Vosniadou, S., & Tsaparlis, G. (August, 2001). The development of students' understanding of water as a solvent. *Paper presented at the Biennial meeting of the European Science Education Association, Thessaloniki, Greece.*

Kuhn, T. (1970). *The Structure of Scientific Revolutions* (2nd edition). Chicago: Chicago Press. First edition: 1962.

Kyrkos, C. & Vosniadou, S. (August, 1997). Mental models of plant nutrition. *Poster presented at the 7th European Conference for Research on Learning and Instruction, Athens, Greece.*

Lakatos, I. (1970). Falsification and the methodology of scientific research programmes. In I. Lakatos & A. Musgrave (Eds.), *Criticism and the growth of knowledge*, (pp. 91-196). Cambridge: Cambridge University Press.

Lipton, J.S., & Spelke, E.S. (2003). Origins of number sense: Large numbers discrimination in human infants. *Psychological Science, 4*, 396-401.

Mahoney, M.S. (1997). Revolution in mathematics. (Unpublished manuscript.)

Megalakaki, O., Ioannides, C., Vosniadou, S., & Tiberghien, A. (August, 1997). Differentiating force from energy: Children's understanding of the concepts of energy and force. *Seventh European Conference for Research on Learning and Instruction, Athens, Greece.*

McCloskey, M. (1983). Naive theories of motion. In D. Gentner & A.L. Stevens (Eds.), *Mental models* (pp. 299-323). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Merenluoto, K., & Lehtinen, E. (2002). Conceptual change in mathematics: Understanding the real numbers. In M. Limon & L. Mason (Eds.), *Reconsidering conceptual change: Issues in theory and practice* (pp. 233-258). Dordrecht: Kluwer Academic Publishers.

Novak, J.D. (1977). An alternative to Piagetian psychology for science and mathematics education. *Science Education, 61*, 453-477.

Pintrich, P. (1999). Motivational beliefs as resources for and constraints on conceptual change. In W. Schnotz, S. Vosniadou & M. Carretero (Eds.), *New perspectives on conceptual change* (pp. 33-50). Killington: Oxford.

Posner, G.J., Strike, K.A., Hewson, P.W., & Gertzog, W.A. (1982). Accommodation of a scientific conception: Towards a theory of conceptual change. *Science Education*, *66*, 211-227.

Schoenfeld, A. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Eds.), *Cognitive science and mathematics education* (pp. 191-215). Hillsdale NJ: Lawrence Erlbaum Associates.

Sfard, A. (1987). Mathematical practices, anomalies and classroom communication problems. In P. Ernest (Ed.) *Constructing mathematical knowledge: Epistemology and mathematics education*. (Studies in Mathematics Education Series, 4). London: The Falmer press.

Smith, J.P., diSessa, A.A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences*, 3, 115-163.

Spada, H. (1994). Conceptual change or multiple representations? *Learning and Instruction, 4*, 113-116.

Stafylidou, S., & Vosniadou, S. (2004). Students' understanding of the numerical value of fractions: A conceptual change approach. . In S. Vosniadou & L. Verschaffel, *The Conceptual Change Approach to Mathematics Learning and Teaching, Special Issue of Learning and Instruction*, 14(5), 503-518.

Stavy, R., & Tirosh, D. (2000). *How students (mis)understand science and mathematics: Intuitive rules*. New York: Teachers College Press.

Tirosh, D., & Tsamir, P. (2004). What can mathematics education gain from the conceptual change approach and what can the conceptual change approach gain from its application to mathematics education? In S. Vosniadou & L. Verschaffel, *The Conceptual Change Approach to Mathematics Learning and Teaching, Special Issue of Learning and Instruction*, 14(5), 535-540.

Vamvakoussi, X., & Vosniadou, S. (2004a). Understanding density: Presuppositions, synthetic models and the effect of the number line. In S.Vosniadou, C.Stathopoulou, X.Vamvakoussi, & N.Mamalougos (Eds.), *4th European Symposium on Conceptual Change: Philosophical, Historical Psychological and Educational Approaches* (pp.98-101). May, 19-23, 2004, Delphi, Greece. Athens: Gutenberg Press.