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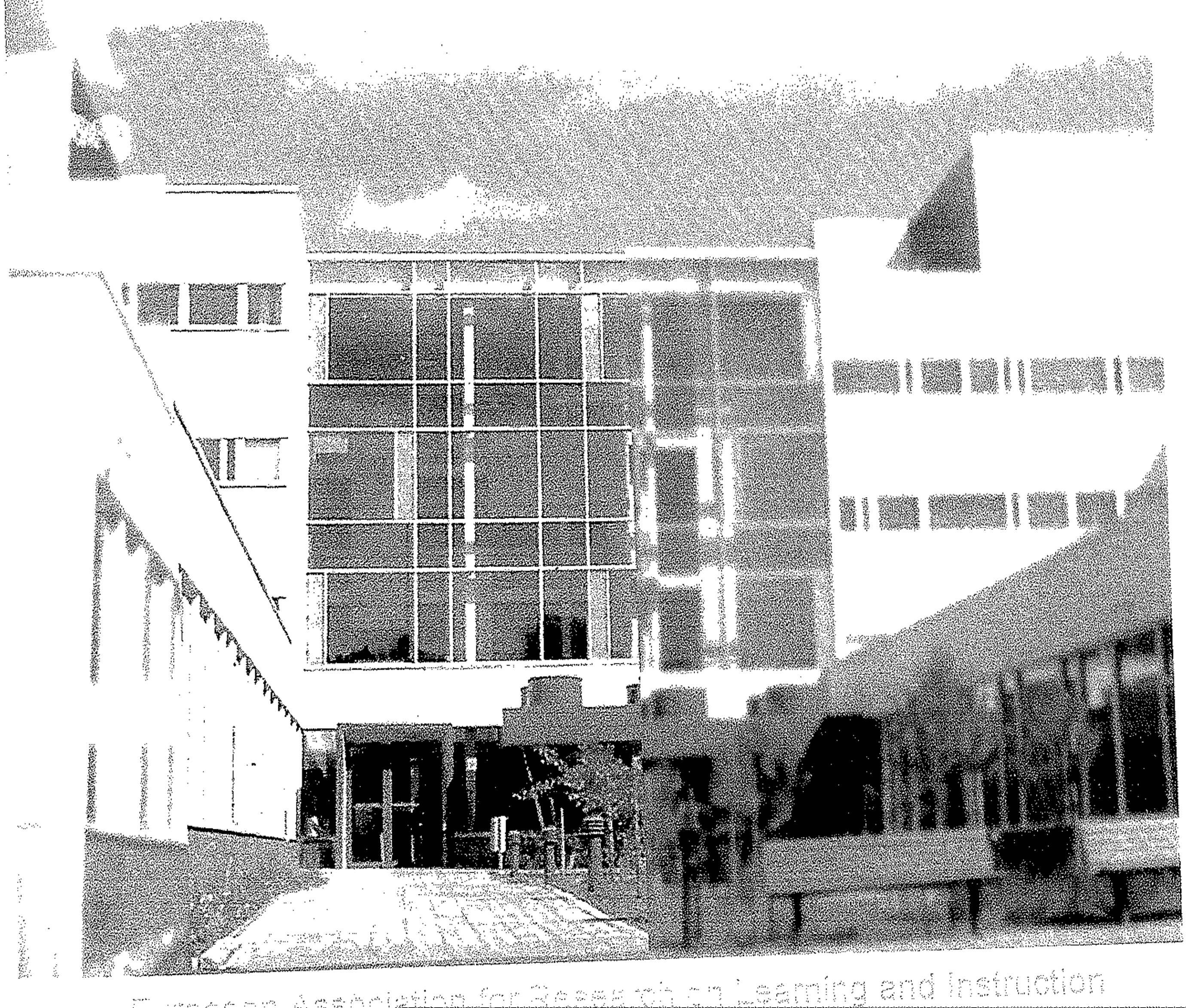

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Conceptual Change in Mathematics: From the Set of Natural to the Set of Rational Numbers

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Abstract

In this study, we investigated ninth grade students' ideas about the structure of the set of rational numbers. The idea of discreteness is considered a fundamental presupposition, which serves as a barrier to understanding the density of the set of rationals. We expected students to generate synthetic models of the set of rational numbers, as well as of the numberline, in their attempts to deal with density related mathematical tasks. We interviewed 16 ninth graders, of various levels of performance. We report our first results, which support our hypothesis

Research shows that prior knowledge about natural numbers stands in the way of understanding rational numbers. For example, properties of natural numbers such as "the more digits a number has, the bigger it is" are used in the case of decimals. Some misconceptions about fractions are connected to principles that apply only in the case of natural numbers (Hartnett & Gelman, 1998). In the context of mathematical operations, there are well known misconceptions, such as that "multiplication always makes bigger", which in this case reflects the effect of prior knowledge about multiplication with natural numbers (Fischbein, Nello & Marino,1985).

In this study, we investigated ninth grade students' ideas about the structure of the set of rational numbers. The history of Mathematics, as well as empirical reasearch, suggests that the conceptual shift from discreteness to density is a demanding one (Lehtinen, Merenluoto & Kasanen, 1997). In terms of the theory of conceptual change that Vosniaodou proposes, the idea of discreteness is considered a fundamental presupposition, which serves as a barrier to understanding the density of the rationals. We expect students to generate synthetic models both of the set of rational numbers and of the numberline, in their attempts to deal with mathematical tasks .(Vosniadou, 1994)

Participants

The participants of the study were 16 ninth graders (approximate age15). All, but one, were members of the same class. We asked three of their teachers -including their mathematics teacher- to help us select students of various levels of performance. The participation was volontary. Only one of the selected students refused to participate and was replaced.

Procedure

All students were interviewed. The interview took place at their school, during the regular school hours. Each interview lasted about 1 hour and covered many aspects of the students' understanding of the number concept, but in this paper we will only refer to the results pertaining to the density of the set of rational numbers. All interviews were recorded and transcribed for the purposes of the present study..

Materials

In a preliminary study (Vamvakoussi, 2001), we had developped a questionnaire to diagnose if and how the tenth grade students' understanding of the concept of number changes, as they acquire expertise on real numbers. It turned out that the tasks regarding the density of rational numbers had some interesting results. We used the same items in the present study. The students were faced with the problem of deciding how many numbers exist between two rational numbers and were asked to find, if possible, the successor of a number in the set of real numbers. The students were asked to think aloud, and comment on their own answers. Additional questions were asked for the purpose of clarification. Finally, we encouraged them to use the numberline to help them deal with the problem at hand and asked them to describe or draw how they pictured the numbers in question on the numberline.

Results

Trying to specify how many rational numbers there are between two given rational numbers, students displayed a strong tendency to group the same kind of numbers together. By this, we don't mean that they group decimals together with decimals and fractions together with fractions, but that they group decimals *with the same number of decimal digits* together and fractions *with the same denominator* together. For example, a participant who was asked how many numbers there are between 0.001 and 0.01 said:

"*...0.01 is equal to 0.010. And we start from 0.001, so in between there are 9 numbers: 0.002, 0.003, up to 0.009*".

When asked, how many numbers there are between 0.005 and 0.006, he said:

"*...There are the numbers 0.0051, 0.0052, up to 0.0059*".

When we explicitly asked him to reconsider both of his answers, he explained that " *I totally forgot about the fourth decimal digit...It didn't occur to me... Somehow, it* (0.0051) *does not seem to belong to the same group* (as 0.010 and 0.001)".

It is rather interesting to notice that, grouping the same kind of numbers together results to sets of numbers that are structured like the set of natural numbers, preserving the property of discreteness. We asked several questions of the type "how many numbers are there between...". Students' answers to three of them are displayed in the following tables.

Table 1.

Students' answers to the question "How many numbers are there between 0.005 and 0.006?"

Student answer	N
There is no other number	9
There are finitely many numbers	3
There are infinitely many (decimal) numbers	3
No answer	1

Table 2.

Students' answers to the question "How many numbers are there between $\frac{3}{8}$ and $\frac{5}{8}$?"

Student answer	N
There is only one number	9
There are "many" numbers, all equal to $\frac{4}{8}$	2
There are finitely many numbers (more than one)	2
I could find infinitely many numbers, if I turned them into decimals	1
there are infinitely many (decimal) numbers	3

Table 3

Students' answers to the question "How many numbers are there between 0.001 and 0.01?"

Student answer	N
There is no other number	3
There are finitely many numbers	10
There are infinitely many (decimal) numbers	3

The results displayed on the tables indicate that students' understandings of the structure of the set of rational numbers are shaped by their understanding of the set of natural numbers.

Some additional remarks, based on our data, may be of some interest, although further analysis is needed:

- Only 1 out of 16 students answered that there are infinitely many numbers[7] between two rational numbers, both in the case of decimals and in the case of fractions. Even this student indicated that there are infinitely many *decimal* numbers between two given decimals and infinitely many *fractions* between two given fractions.
- 1 out of 3 students who answered that there are infinitely many decimal numbers betweeen two given decimals, claimed that there are finitely many fractions, between two given fractions (with the same denominator)

[7] It is not implied here that the student used this expression.

- 1 student out of three students who answered that there are infinitely many fractions, between two given fractions (with the same denominator), claimed that there are finitely many decimal numbers between two given decimals. This student was, according to all three of the teachers who help us select our sample, the most competent student of the whole school.
- All sudents who answered that there are finitely many numbers, between two given rational numbers, explicitly described sets of decimal/ fractional numbers with the discrete structure of natural numbers.
- 8 out of 16 students actually defined the "successor" of a given real number.
- Only 2 of the students altered their initial answers, after they were asked to use the numberline. In both cases, the numberline helped them generate a more sophisticated, yet discrete, model.
- Descriptions and drawings of the numberline clearly reflect the effect of the idea of discreteness.

Discussion

Understanding of rational numbers is a demanding task. We focused on the learning difficulties pertaining to the property of density of rational numbers. The results support our hypothesis, that understanding of the structure of the set of natural numbers stands in the way of understanding the property of density. The ninth graders who participated in our study, by the time we interviewed them, had finished a review of everything they are supposed to know about real numbers, including operations, ordering and turning a fraction into a decimal and vice versa. They have also used the numberline extensively, both as a way to represent the real numbers and as a tool, when solving inequations. Although their procedural knowledge in the area was adequate, our data indicate that they had not developed any sense of homogeneity of different kinds of rational numbers. Moreover, the mental models they generated when trying to deal with mathematical tasks that demand, implicitly or explicitly, knowledge about the structure of the rational numbers, were constrained by the presupposition of discreteness. The same presupposition seems to constrain the understanding of an external represantation of real numbers, namely the numberline. With a first analysis of our data, we traced different types of generic mental models about the numberline, constrained by the pressuposition of discreteness, as well as the use of the ruler. Further analysis is needed towards this direction.

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