REVEALING MODES OF KNOWING ABOUT DENSITY

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Rational number density has been investigated through open-ended question tasks and multiple-choice tasks or by asking to interpolate a number between two numbers. However, students' responses to these three types of tasks were not directly compared. The objective is to look for relationships between the three types of tasks in order to identify differences regarding students' density understanding depending on the type of knowledge elicited. Participants were 791 primary and secondary school students. Results show that most of the students believed that rational numbers are discrete. Differences between the modes of representation are also found. Finally, interpolating a number between two pseudo-consecutive ones is neither a necessary nor a sufficient condition for students to answer that there are infinitely many intermediate numbers.

THEORETICAL AND EMPIRICAL BACKGROUND

Understanding the density of rational numbers is considered a stumbling block for primary and secondary school students (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004), and even for undergraduates (Tirosh et al., 1999). Most of the difficulties have been attributed to the interference of the natural number based prior knowledge (Alibali & Sidney, 2015; Smith et al., 2005). While the natural number set is discrete (between two numbers there is a finite -possibly zero- number of numbers), the rational number set is dense (there is an infinite number of numbers between any two rational numbers).

Ample studies showed that the idea of discreteness is a "fundamental presupposition which constrains students' understanding of the structure of the set of rational numbers" (Vamvakoussi & Vosniadou, 2004, p. 457). Students believe that between two rational numbers there is only a finite (including zero) number of intermediate numbers. For instance, students believe that between the "pseudo-consecutive" fractions 5/7 and 6/7 there are no numbers, or that between 1/2 and 1/4 there is only one number, 1/3 (Merenluoto & Lehtinen, 2004). In decimal numbers, students think that there are no numbers between the "pseudo-consecutive" numbers 0.59 and 0.60, or that only 1.23 is between 1.22 and 1.24 (Moss & Case, 1999). Furthermore, students sometimes treat fractions and decimals numbers as unrelated sets of numbers, rather than as interchangeable representations of the same number (Khoury & Zazkis, 1994). Vamvakoussi and Vosniadou (2010) showed that some students believe that there are

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only decimals numbers between two decimals numbers and fractions between two fractions.

Previous studies by Vamvakoussi and Vosniadou have identified intermediate stages in secondary school students' density understanding. Some of these studies used open-ended question items (e.g., Vamvakoussi & Vosniadou, 2004). This type of task taps on students' available resources, at least the ones that they employ spontaneously (i.e., without any guidance). For instance, students who rely heavily on natural number knowledge might treat the given numbers as endpoints of a segment of the natural number sequence (responding that there is no other number between 2.1 and 2.2, or only three numbers between 2.1 and 2.5), or students might be able to interpolate more numbers, using transformation strategies, such as converting 2.1 and 2.2 to 2.10 and 2.20, respectively and they might or might not realize that this process is repeatable, thereby coming to understand density. It is also possible that some students have some experience with similar tasks and recall the correct answer ("there are infinitely many numbers in between"). Other studies used multiple-choice items (e.g., Vamvakoussi & Vosniadou, 2010). This type of task is typically facilitating for students, because they present the correct answer among a number of "naïve" and "less naïve" answers. This might prompt students to think of more sophisticated strategies or recall the correct answer. Finally, other studies asked students to interpolate (write) a number between two "pseudo-consecutive" numbers (e.g., Van Hoof et al., 2015). Such items require students to show whether they do have a strategy available to produce one number between two "pseudo-consecutive" numbers, if specifically asked to. Such strategies arguably constitute a possible necessary first step in the process of understanding the infinity of intermediates in an interval.

Although all these three types of tasks have been used to investigate students' understanding of density, to the best of our knowledge, no study has directly compared students' responses across tasks. This is the purpose of this study: We aimed at investigating whether the task format (open-ended or multiple-choice) makes a difference in students' responses; and whether being able to interpolate one number between two (pseudo-consecutive) ones is indeed a necessary condition for students to answer that there are infinitely many intermediates in an interval.

Identifying various ways of (not or not completely) understanding density

The current research aims at identifying differences in density understanding, by using the aforementioned variety of task types. It is part of a larger quantitative study with 953 Spanish primary and secondary school students.

The students in our sample answered a paper-and-pencil test composed of 13 items (González-Forte et al., 2022): three write items, six question items, and four multiple-choice items. In *write items* students had to write a number between two given rational numbers (between 3.49 and 3.50; 1/3 and 2/3; 1/8 and 1/9). In *question items* students had to answer the question how many numbers there are between two given rational numbers (1.42 and 1.43; 1.9 and 1.40; 2.3 and 2.6; 2/5 and 3/5; 2/5 and 4/5; 5/9 and

5/6). In *multiple-choice items* students had to answer the question how many numbers there are between two given rational numbers (3.72 and 3.73; 0.7 and 0.9; 1/3 and 2/3; 1/6 and 4/6), choosing one out of the seven answers offered, including the correct answer. Multiple-choice items were always at the end of each test since the word "infinite" appears and can help them to correctly solve the other items

Since the three types of tasks may elicit different response profiles in students, a cluster analysis per task was performed (González-Forte et al., 2022). In the write items, six profiles were identified: Students who considered that it was impossible to write a number between two pseudo-consecutive numbers (called *Naïve*). Students who considered that it was impossible to write a number between two pseudo-consecutive decimal numbers, but in fractions answered in a naïve consecutive way (i.e., 1/4 is between 1/3 and 2/3) (*Fraction consecutive*). Students who correctly wrote a number between two pseudo-consecutive decimals, and in fractions i) considered that it was impossible to write a number between two pseudo-consecutive fractions (*Correct decimals fraction naïve*); ii) answered in a naïve consecutive way (*Correct decimals fraction consecutive*); iii) correctly wrote a number between two pseudo-consecutive fractions with the same denominator, but in fractions with the same numerator, they considered that it was impossible to write a number (*Almost correct*). Students who correctly wrote a number between two pseudo-consecutive numbers (*Correct*).

In the question items, seven profiles were identified: Students who considered that there was no other number between two pseudo-consecutive numbers, and that there was a finite number of numbers between two non-pseudo-consecutive numbers (*Naïve*). Students who considered that there was no other number between two pseudo-consecutive fractions, and between two pseudo and non-pseudo-consecutive decimals i) considered that there was a finite number of numbers (*Decimal finiters*), ii) calculated the difference (*Decimal differencers*), iii) considered that there was an infinite number of numbers (*Correct decimals fraction naïve*). Students who considered that there was a finite number of numbers between two pseudo and non-pseudo-consecutive decimals and fractions (*Finiters*). Students who considered that there was an infinite number of numbers between two different fractions and two different decimals (*Correct*). Students with a generally low performance in all items who provided answers without any pattern (*Rest*).

In the multiple-choice items, nine profiles were identified: Students who considered that there were no numbers between two pseudo-consecutive numbers, and that there was a finite number of numbers between two non-pseudo-consecutive numbers (*Naïve*). Students who considered that there were no numbers between two pseudo-consecutive decimals, and a finite number of decimals between two non-pseudo-consecutive decimals, but in fractions considered that there was a finite number of fractions between two different fractions (*Decimal naïve fraction finiters*). Students who considered that there was a finite number of numbers between two different fractions and two different decimal numbers (*Finiters*). Students who considered that there was an infinite number of decimals between two different decimal numbers, and between two different fractions i) considered that there were no numbers between two pseudo-consecutive fractions, and a finite number of fractions between two non-pseudo-consecutive fractions (*Decimal infiniters fraction naïve*), ii) considered that there was an infinite number of decimals (*Decimal infiniters*), iii) considered that there was an infinite number of fractions (*Infiniters*), iv) considered that there was an infinite number of numbers that could be represented by several different representations, such as decimals and fractions (*Decimal infiniters correct fractions*). Students who considered that between two different fractions and two different decimals there was an infinite number of numbers that can be represented by several different representations, such as decimals and fractions (*Correct*). Students with a generally low performance in all items who provided answers without any pattern (*Rest*).

RESEARCH GOAL

The three types of tasks elicit different knowledge about density: actively producing the statement that "there are infinite numbers" (*question items*), recognizing the correct answer (*multiple-choice items*) and having procedures to find an intermediate number (perhaps while knowing that there are infinitely many) (*write items*). However, so far students' responses to these three types of tasks were not directly compared. Therefore, the aim of this study is to look for relationships between the three types of tasks in order to identify differences regarding students' density understanding depending on the type of knowledge elicited.

METHOD

The final sample on which this study is based consists of 791 primary (5th and 6th grade) and secondary school students (from 7th to 10th grade). We did not include the 162 students who belonged to the "Rest" profile in question items (see above), as they performed so low in these items and their responses lacked any pattern, which made us conclude that it would be difficult and not meaningful to identify relationships with respect their performance on the other item types.

To compare among tasks, we categorized the profiles in broader categories looking for common characteristics (see Table 1). First, we categorized the profiles obtained in question and multiple-choice items looking at students' consistency in providing "infinitely many intermediates" or "a finite number, possibly zero, of intermediates" answers across items. Four categories were identified: *FIN*: students who consistently answered that there is a finite number (possibly zero) of intermediate numbers within task. *D-INF/F-FIN*: students who consistently answered that there are infinitely many intermediates between decimals, but a finite number between fractions. *FIN/INF*: students who provided "finite" and "infinite" responses within task, without following any recognizable pattern with respect to the type of the endpoints (i.e., decimals, or fractions). *INF*: students who consistently responded that there are infinitely many intermediates within task.

As far as the write items is concerned, we categorized profiles in four possible categories: *D & F correct*: to correctly interpolate a number between decimals as well as fractions. *Only D correct*: to correctly interpolate a number only between decimals. *Only F correct*: to correctly interpolate a number only between fractions. This alternative was not present in our findings. *D & F none or incorrect*: to incorrectly interpolate a number, or not to be able to interpolate at all (by answering "impossible"), for decimals as well as for fractions.

Item	Profile	Category
Question	Naïve Decimal differencers Decimal finiters Finiters	FIN
	Correct decimals fraction naïve	D-INF/F-FIN
	Correct	INF
Multiple-choice	Naïve Decimal naïve fraction finiters Finiters	FIN
	Decimal infiniters fraction naïve	D-INF/F-FIN
	Rest	FIN/INF
	Infiniters Decimal infiniters Decimal infiniters correct fractions Correct	INF
Write	Naïve Fraction consecutive	D & F none or incorrect
	Correct decimals fraction naïve Correct decimals fraction consecutive Almost correct	Only D correct
	Correct	D & F correct

Table 1: Broader categories identified

RESULTS

Table 2 presents the four general groups identified.

Table 2: Summary of the four general groups identified

Item		Group 1	2	3	4
Question		FIN	FIN	D-INF, F-FIN	INF
Multiple-choice		FIN	FIN/INF	FIN/INF	INF
Write	D & F correct	4 (1.62%)	8 (2.95%)	2 (2.56%)	56 (28.72%)
	Only D correct	72 (29.15%)	127 (46.86%)	71 (91.03%)	131 (67.18%)
	D & F none or incorrect	171 (69.23%)	136 (50.18%)	5 (6.41%)	8 (4.10%)
	Total	247	271	78	195

Group 1: Naïve (*n* = 247, 31.2%): Students who provided FIN answers consistently across multiple-choice and question items. The great majority of students in this group could not interpolate a number, neither between decimals nor between fractions. About one third of the students correctly interpolated a number between decimals; and four students correctly interpolated a number between decimals and fractions.

Group 2: Question items-Finiters (*n* = 271, 34.3%): Students who consistently provided FIN answers in question items and provided some "infinitely many intermediates" answers in some, but not all multiple-choice items. This group was better in interpolating (at least between decimals) than Group 1 students – still half of them were not able to interpolate neither between decimals, nor between fractions.

Group 3: Question items-Decimal Infiniters (*n* = 78, 9.9%): Students who gave INF answers for decimals but FIN answers for fractions in question items. Some of these students retained this response pattern in multiple-choice items, while others provided a mixture of FIN and INF answers. The great majority of Group 3 students were able to interpolate a number between decimals, but not between fractions.

Group 4: Advanced (*n* = 195, 24.6%): Students who provided INF answers in question items, and the majority of students (*n* = 178) also gave INF answers in all multiple-choice items. Within this (advanced) group, there are two interesting subgroups depending on how they solved the write items:

- **Group 4.1: Write items-Infiniters** (*n* = 56): Students who provided correct answers both in question and write items. These students answered that there are infinitely many numbers between two different fractions and two different decimal numbers and were able to correctly interpolate a number between them.
- **Group 4.2: Write items-Decimal infiniters** (*n* = 139): Students who provided correct answers in question items, but in write items, the majority (*n* = 131) could interpolate a number only between decimals. There are 8 students who, despite answering consistently that there are "infinitely many intermediates" in question as well as multiple-choice items, answered "impossible" in the write items.

DISCUSSION AND CONCLUSIONS

Comparing students' responses across three different tasks has provided some interesting findings beyond those found through assigning students to different profiles per task (González-Forte et al., 2022). Firstly, we could induce the existence of four different groups of students according to the way they have answered the three different tasks. A first major finding when looking at these groups' characteristics, is that the great majority of students consistently provided FIN answers in question items (Groups 1 and 2: n = 518). That is, when being asked in an open question how many numbers there are between two different fractions or decimals numbers, most of the primary and secondary school students answered that there is a finite (possibly zero) number of intermediates. There was a big group of students (Group 1, n = 247) who retained this response pattern also in the multiple-choice items, indicating that they were confident enough with their response to ignore the presence of more sophisticated options. A somewhat larger group (Group 2, *n* = 271) provided (some) "infinitely many intermediates" answers in the multiple-choice task, indicating that they recognized or recalled the correct answer. Group 1 and Group 2 students performed generally poorly in the write task. Still about one third and half of the students in Group 1 and 2, respectively, were able to interpolate a number between decimals, apparently without realizing that this process is repeatable, which could give them the insight about density.

It should be noted that being more competent with interpolating a number between decimals than between fractions was present across groups. This is clearly evident for Group 3 (*n* = 78). Interestingly, these students consistently answered "infinitely many intermediates" for decimals, but "a finite number or intermediates" for fractions in question items. Half of them persisted in answering the same in the multiple-choice task, despite having the correct answer as an option. It is possible that some students of Group 3 had realized that the process of interpolation can be repeated ad infinitum for decimals. In any case, these students had not realized that decimals and fractions are different representations of the same numbers, rather than different numbers (Vamvakoussi & Vosniadou, 2010).

Finally, students in Group 4 (*n* = 195) consistently answered that "there are infinitely many intermediates" in question and multiple-choice tasks and they also performed better in the write tasks, compared to all previous groups. Still, only a subgroup of these students (Group 4.1, *n* = 56) was able to interpolate a number between decimals as well as fractions.

It thus appears that the competence to interpolate one number between two given (pseudo-consecutive) ones is neither a necessary nor a sufficient condition for students to answer that there are infinitely many intermediate numbers (or choose this answer). The latter is not surprising, since interpolating one number does not lead to the realization that there are, in fact, infinitely many numbers, unless one also realizes that this process is repeatable. The interesting finding is that students may provide "infinitely many" answers and still not have the necessary competencies to produce intermediate numbers. This could indicate that they merely recall (in open-ended tasks) or recognize (in multiple-choice tasks) the correct answer. Therefore, we should not overinterpret what learners really understand about density if they respond correctly to multiple-choice items or question items (items often used in previous research), as in other types of items that elicit other knowledge, they still show a lack of understanding.

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