Teaching fraction multiplication and fraction division to prospective teachers: A design research study

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We present findings from the first cycle of an ongoing topic-specific design research study focused on fraction multiplication and division. We designed a course unit addressed to prospective teachers to support them in building productive meanings for multiplication and division that are transferable to rational number contexts. Thirty-four prospective teachers participated in the first enactment of the course unit. They were pre- and post-tested via a paper-and-pencil test comprising 11 tasks, examining various aspects of knowledge in this area that are necessary for teaching. The results showed that their performance improved significantly after the intervention, but substantial conceptual difficulties were identified at both test moments.

Keywords*: Fraction multiplication, fraction division, conceptual change, teacher education.*

Introduction

Multiplication and division of fractions present teachers with many challenges (Copur-Gencturk, 2021; Izsák, 2008; Ma, 1999). Typically, teachers are able to perform computations, but face difficulties in problem solving, problem posing as well as in representing and interpreting the two operations and their algorithms (Ma, 1999; Olanoff et al., 2014). In terms of the theoretical framework of Ball and colleagues (e.g., Ball et al., 2008), such evidence indicates that teachers lack in content knowledge necessary for teaching mathematics, either Common Content Knowledge (i.e., mathematical knowledge necessary for problem solving in various contexts, including teaching; hereafter CCK); or Specialized Content Knowledge (i.e., knowledge that is used specifically in the context of teaching and allows, for example, for explaining, exemplifying, and representing mathematical content for students; hereafter SCK). Such limitations are bound to have adverse effects on the quality of teaching and, consequently, on students' learning.

In this paper we present findings from the first cycle of a topic-specific design research study (Gravemeijer & Prediger, 2019) aimed at developing materials and teaching-learning arrangements to enhance pre- and in-service teachers CCK and SCK on fraction multiplication and division.

Theoretical Framework

A major source of conceptual difficulties with fraction multiplication and division is the changes in meaning required in the transition from natural to rational numbers. The dominant meanings of these operations within natural numbers—namely "repeated addition" and "fair sharing"—do not generalize for fractions (Fischbein et al., 1985; Tirosh, 2000). In addition, the relationship between multiplication and division (as inverse operations) is reframed via the introduction of reciprocals, allowing for division to be replaced by multiplication by the reciprocal of the divisor. This reframing is crucial as it underpins the fraction division algorithm commonly known as "invert and multiply".

It is widely known that these changes are a source of difficulty for students as well as for pre-and in service teachers (Ma, 1999; Olanoff & Tobias, 2014; Tirosh, 2000). This phenomenon can be theoretically approached by conceptual change perspectives on learning that is focused on problems faced by learners when their prior knowledge in an area is incompatible with the new content to be learned; we thus draw systematically on principles for instruction stemming from conceptual change approaches to mathematics learning, in particular rational number learning (Vamvakoussi, 2017) to inform our design. More specifically, in the phase of prospective analysis (Gravemeijer & Prediger, 2019) of our study, we looked for productive meanings for multiplication and division that are transferrable from natural to rational numbers (see also Simon et al., 2018). To this end, we invested in a relational approach to multiplication, interpreting an expression of the form a x b=c as depicting a relationship between three physical and/or numerical quantities (Polotskaia & Savard, 2021). For example, the expression 3x5 =15 is typically interpreted as "3 times 5 is 15"; from a relational perspective, it could be interpreted as "the triple of 5 is 15" and, simultaneously, "5 is one-third of 15". This is the same multiplicative relationship viewed from two perspectives, that allows for assigning a similar role to the natural and fractional number as multiplicative operators. The multiplicative relation between similar quantities can be established through measurement, a notion crucial also for the operation of division, since this meaning is transferable to rational numbers. A meaning for multiplication that is also transferable to rational numbers draws on the "rectangular area" model (Iszák, 2008), where the factors are represented as the lengths of the sides of a rectangle and the product as its area. The inverse problem situation, where the length of one side is the unknown quantity, affords a viable meaning for fraction division as well (Yim, 2010).

To foster the intended meanings for multiplication and division, we designed a course unit targeting prospective elementary teachers, consisting of short lectures, in-class activities, and self-study tasks. Consistent with our theoretical lenses (Vamvakoussi, 2017), the key ideas underlying the design of all components of the course unit were the following: a) explicitly problematizing the meaning of multiplication and division already in the context of natural numbers; b) systematic cross-domain mapping (i.e., an analogy between dissimilar domains) between the domain of physical quantities, specifically length and area, and the domain of number, established through measurement and implemented in terms of the representations as well as the problem situations employed; c) highlighting (deep) similarities and differences between natural and rational number contexts by systematically varying the types of numbers in the same problem situations (see also Sun, 2017). All instructional tasks for students were framed as classroom scenarios (Biza et al., 2007), with a view to making the tasks relevant to their professional interests. Students were asked to evaluate problems and problem solutions provided by hypothetical students, compare different representations, provide feedback, solve as well pose problems related to fraction multiplication and division; and to evaluate and modify instructional tasks. The course unit was enacted with prospective teachers. In this paper, we only present findings regarding students' performance in fraction multiplication and division tasks before and after the intervention. Data collected during the intervention are currently under analysis and will inform the retrospective analysis of the first implementation.

Method

The participants were 34 prospective teachers (hereafter, PTs) from a Department of Primary Education in Western Greece (26 female). According to their Study Program (8 semesters in total), the PTs completed two compulsory courses related to elementary mathematics and mathematics education in the primary years (Grades 1 to 6). The participants attended a 7th semester elective course on mathematics education, within which the unit was embedded. They gave their informed consent for their research data to be used anonymously for research purposes.

We designed a test consisting of 11 tasks (T1 - T11) focusing on fraction multiplication and division. Except for 3 of the tasks (T1, T10, T11), all were framed as classroom scenarios (Biza et al., 2007). The PTs were presented with hypothetical students' responses to mathematical tasks in the form of solutions, representations, or statements, and were asked to evaluate them in terms of mathematical accuracy; to explain students' thinking; and to provide feedback, if necessary. In this paper, we will refer only to PTs' evaluations which reflect their CCK (T1, T2, T5, T6, T7) or SCK (T3, T4, T8, T9, T10, T11; Ball et al., 2008). Specifically, the first group of tasks examine whether the PTs can themselves perform fraction multiplication and division; recognize the appropriate operation for solving a problem; and can exchange dividing by n with multiplying by 1/n (CCK). The latter group of tasks examine whether PTs can correctly evaluate alternative solution processes, verbalize a rule accurately, and exemplify fraction multiplication and division via word problems or non-symbolic representations, which are actions that are particular to teaching (SCK).

In tasks T1 and T2, the PTs were asked to calculate the results of the operations 2/5×7/3 and 2/5: 7/3, respectively, writing down the calculation process in detail. In T3, the PTs were asked to evaluate the following calculation process: 3/4×5/2 = 3/4×10/4 = 30/16. If a PT acknowledged that the calculation process is correct despite the fact that converting to equivalent fractions is unnecessary, the response was deemed correct. In T4, the PTs were asked to evaluate the statement, "When I want to divide two fractions, I invert one and multiply". If a PT acknowledged that it matters which of the two fractions is inverted, the response was deemed correct. In T5, the PTs were asked whether the problem "5 kilograms of cherries cost 10 euros. How much does half a kilogram cost?" reflects the operation 5:1/2. Similarly, in T6, they were asked to judge whether the problem "A carpenter measures the length of a plank with a stick and finds it to be 18 sticks long. What would he find if he measured the length of the plank with two-thirds of this stick?" can be solved via the operation 2/3x18.

In T7, a circular diagram (Figure 1) was presented as a task given to students to fill in the missing number, and the PTs had to evaluate a statement claiming that this diagram is incorrect. This task examines whether PTs could substitute division with multiplication by the reciprocal of the divisor. Responses stating the diagram is correct and/or mentioning the missing number were deemed correct. In T8, the PTs were asked whether the representation in Figure 2 is a valid representation of the operation 2/3×3/4. This representation is, according to Iszak (2008), a powerful representation for fraction multiplication. In T9, the PTs were asked to create their representation for this operation.

Finally, in tasks T10 and T11, the PTs had to pose a problem matching the operations 3/4×2/5 and 4: 2/3, respectively. In these tasks, we examined which types of problems were created and whether they were appropriate (i.e., they matched the operation in question and used the given numbers).

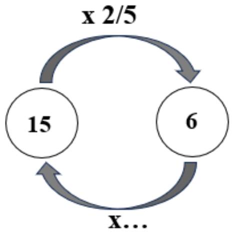

Figure 1: Circular diagram (T7)

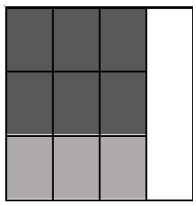

Figure 2: Fraction multiplication representation (T8)

The PTs completed the test before the intervention. The course unit was enacted in 6 sessions, approximately 2 hours each. The PTs worked on the in-class activities in groups of 3-5 members. Three research team members (first, third, and last author) were present to support student work and document their difficulties. The last author provided feedback in the form of short interactive lectures. After the intervention, the pretest sheets were returned to the participants who had the opportunity to review their initial responses and revise them if they judged it necessary. An identical empty sheet was provided to fill in the revised responses. Posttest responses included the unrevised and the revised responses.

Results

Responses were coded as correct/incorrect and scored by 0 and 1, respectively. The total score was calculated as the sum of correct responses across tasks. The Shapiro-Wilk test (sample size<50) showed that the data tested were normally distributed (p=>.05). The paired sample t-test showed a statistically significant difference between the pre- and post-test scores, $t(33) = -6.849$, $p = .000$. Specifically, there was a significant increase in the mean scores from pretest (M=4.76, SD =1.759) to posttest (M=7.35, SD=1.454). Table 1 presents the percentages of correct responses per task in the pre- and post-test. In all tasks, an increase in the percentage of correct answers is observed at post-test, with tasks T7, T8, T9, and T10 showing the most substantial improvement.

Table 1: Percentage of correct answers per task, in the pre- and post-test

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11
Pre-test	85,3%	82,4%	35,3%	64,7%	82,4%	41,2%	44,1%	17,6%	0	11,8%	11,8%
Post-test	97,1%	94,1%	44,1%	73,5%	88,2%	61,8%	85,3%	61,8%	52,9%	35,3%	35,3%

Table 1 shows that the PTs reached their highest scores in T1 and T2 (i.e., in the calculation tasks), at both test moments. However, evaluating a calculation process for fraction multiplication (T3) as well as an inaccurate expression of the “invert and multiply” rule (T4) proved more challenging. Further, it appears that at pretest the majority of PTs recognized that the given operation did not match the given problem in the case of division (T5), but not in the case of multiplication (T6). At posttest, there were still about 40% incorrect responses in T6.

T7 elicited a substantial number of incorrect responses at pretest. The PTs stated that the second operation in the diagram should be division, or that the second operand should be the same as the first; or they explicitly claimed that it is impossible to find a number that, when multiplied by 6, results in 25. Despite the large increase in correct responses at posttest, there were still about 40% incorrect ones in this task. In Task 8, a small number of correct evaluations were made at pretest that increased substantially at posttest. T9 elicited only incorrect responses at pretest; about half of the responses were correct at posttest. In most of the responses deemed incorrect (23 and 12 at pre- and post-test, respectively), the PTs calculated and represented the product (1/2), indicating that they misinterpreted the task. Two PTs represented the given fractions separately and put the multiplication sign in between. Appropriate representations were either grounded on the "part of a part" meaning of multiplication, or the rectangular area model, similarly to T8.

T10 and T11 proved challenging at both test moments. Fifteen PTs at pre-, and 6 PTs at pos-test did not pose any problem for T10 (and were scored by 0). The number of inappropriate problems at pretest was 15 and 3 of them were meaningless or unsolvable. The remaining 8 problems had a common feature: The given fractions were identified as parts of a quantity but were not combined multiplicatively, resulting either in addition problems (e.g., *Paul bought 3/4 of a kilogram of coffee, and Maria gave him another 2/5 of a kilogram. How many kilograms of coffee does Paul have in total?*); or fraction comparison problems (e.g., *John ate 3/4 of a pizza, and Helen ate 2/5. Who ate more*?). At posttest, there were 9 such problems in a total of 15 inappropriate ones. On the other hand, the number of appropriate problems increased from 4 to 13 after the intervention. Two problems at pretest and 5 at posttest were grounded on the rectangular area model of fraction multiplication (e.g., *The length of a rectangle is 3/4 m, and its width is 2/5 m. How much is its area?)*. The remaining were grounded on the "part of a part" meaning (e.g., *Konstantina bought 3/4 of a kilogram of apples and gave 2/5 of that to her friend Maria. How many kilograms did Maria get?*).

In T11, 16 PTs at pretest, and 3 PTs at posttest did not pose any problem. The number of inappropriate problems increased from 14 to 19 after the intervention. In many of these problems (6 at pretest, 12 at posttest), the given fraction (2/3) functioned as a multiplier of a quantity. Some of these problems (4 at pretest, 10 at posttest), were multiplication problems (e.g., *A kilogram of cherries costs 4€. How much do 2/3 of a kilogram of cherries cost?*). In the remaining such problems, arbitrary data were introduced to pose problems that did not correspond either to multiplication or to division with the given numbers, or they were unsolvable; however, 2/3 still functioned as a multiplier of a quantity (e.g., *4 kilograms of potatoes cost €5. How much do 2/3 of a kilogram cost*?). In one problem (at pretest), the roles of 4 and 2/3 were reversed, with 2/3 functioning as the dividend (*4 crackers cover 2/3 of a small baking tray's surface. How many centimeters does 1 cracker cover?*). The remaining inappropriate problems were mostly meaningless or unsolvable; corresponded to other operations, particularly subtraction; or a combination of the above (e.g., *I have 4 liters of milk, and I want to distribute them into 2/3 of a cup. How many liters of milk will I have left?*).

On the other hand, the number of appropriate problems increased from 4 to 12 after the intervention. Most of them were grounded on the measurement meaning of division (e.g., *A baker distributes 4 kilograms of dough into containers with a capacity of one kilogram, but he uses only 2/3 of the capacity. How many containers will he need?*). Only one problem at posttest was grounded on the rectangular area model (*I have a rectangle with an area of 4 square meters. If its length is 2/3 meters, what is its width?*). Additionally, 2 participants posed problems of the type "known part, unknown whole" at pretest, and retained them at posttest (e.g., *If 2/3kg of lemons costs €1, find the cost of 4kg*).

We cross-tabulated the individual total scores (max 11) at pretest with the corresponding ones at posttest (Table 2). Each cell of Table 2 represents the number of participants reaching the particular scores at pretest and posttest. The individual scores range from 1 to 9 at pretest; and from 4 to 11 at posttest, pointing to inter-individual differences with respect to performance at both test moments.

Table 2: Cross-tabulating the individual total scores at pre- and post-test

	Individual total score -Posttest												
Individual total score-Pretest	0	1	2	3	4	5	6	7	8	9	10	11	Total
0							1						1
1							2		1				3
2													
3						1		1	4				6
4							1	2					3
5					1	2	1	3		2	1		10
6							1	1		2		1	5
7							1	1	2				4
8									1				1
9											1		1
10													
11													
Total					1	3	7	8	8	4	2	1	34

Most participants (27/34) improved their score at posttest. The score of 14 of them increased by 1 to 3 points; the score of the other 13 increased by 4 to 6 points. Five participants retained the same score; two did not revise any of their initial responses; the others revised incorrect as well as correct responses. Finally, 2 participants received a lower score at posttest. Further, there were two participants who had a relatively high score already at pretest (8 and 9); one of them was among the ones who retained the same score, and the other increased the score by 1 at posttest. Finally, only one participant reached the maximum score (11), with an initial score of 6.

Conclusions - Discussion

We presented findings from the first cycle of a topic-specific design research study (Gravemeijer & Prediger, 2019) focused on fraction multiplication and division. We designed a course unit addressed to pre-service teachers, aiming to enhance their CCK and SCK (Ball et al., 2008). We targeted the challenges that relate to the transition from natural to rational numbers, thus the design of the course unit as well as the intervention was based on principles from instruction stemming from conceptual change perspectives on learning (Vamvakoussi, 2017). The key idea was to foster productive meaning for multiplication and division already in the context of natural numbers which can be transferred to rational number contexts (see also Simon et al., 2018). In this paper we focused on results regarding the participants' performance before and after the intervention in related tasks.

Consistently with prior research, before the intervention, most participants were able to execute the algorithms of fraction multiplication and division but had very poor conceptual understanding of the two operations and their relationship (Ma, 1999; Olanoff et al., 2004; Tirosh, 2000). This is particularly evident in problem-solving and problem-posing tasks in which, similarly to students, the participants appeared to confuse fraction multiplication with fraction division, fraction multiplication with addition, and fraction division with subtraction. Evaluating and constructing representations pertaining to fraction multiplication and division also proved very challenging (Iszák, 2008). Further, the participants were mostly unaware of the possibility to replace multiplication with division by the reciprocal of the divisor (Ma, 1999), with some of them stating explicitly that it is not possible to multiply a number and get a smaller result (Fischbein et al., 1983, Tirosh, 2000). Finally, a correct, even if atypical, calculation process for multiplication as well as an inaccurate formulation of the "invert and multiply" rule was evaluated incorrectly to a large extent. These results depict limitations in various aspects of Common Content and Specialized Content Knowledge (Ball et al., 2008), and underly the need for substantial support to prospective teachers. After the intervention, the participants' performance increased significantly. However, this should not be taken to imply that they have mastered the intended content. Indeed, many of the difficulties identified by the pretest, persisted after the intervention. This is not uncommon in conceptual change research because revising prior knowledge and building new knowledge in a complex area such as fractions is a challenging and time-consuming endeavor.

Our findings also indicate that there were inter-individual differences with respect to the participants' performance before and after the intervention, as well as to the extent they benefited by the intervention. Analysis of the data collected during the intervention will shed more light on the conditions that favor or constraint learning and will be examined in the phase of retrospective analysis of the study (Gravemeijer & Prediger, 2019) to inform our design for future implementations.

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